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ON GENERALIZED SLANT HANKEL OPERATORS IN THE CALKIN ALGEBRA

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Abstract. In the paper, we introduce and analyze the notion of essentially λ- slant Hankel operator and essentially (λ, μ)- slant Hankel operator on the Lebesgue space $L^2(\mathbb{T})$. Certain basic properties of these operators, connection between them and their co-relations with certain existing operators are also discussed.

1. Introduction

In the beginning of 20th century, O. Toeplitz introduced the notion of a Toeplitz operator on the Hardy space $H^2(\mathbb{D})$. Over the time, mathematicians (namely, Ho [15, 16], Avendaño [3, 4], Barria and Halmos [5]) introduced different generalizations of Toeplitz operators. Due to its applications [6, 20, 21] in different areas (namely, probability, statistics, oscillation and signal processing etc.), the essential commutant of U has been an extensive area of research from many years, where U is the unilateral forward shift operator on $H^2(\mathbb{D})$. The essential commutant of the forward shift operators is known as essentially Toeplitz operators. Barria and Halmos [5] paid more attention to Toeplitz operators and characterized it completely.

The Hankel operators are one of the famous companions of Toeplitz operators. A Hankel matrix (a_{ij}) is a square matrix which is constant on each diagonal orthogonal to the main diagonal. It is finite or infinite matrix with complex entries. The name Hankel matrix is given after the name of famous mathematician H. Hankel. Each Hankel matrix represents a Hankel operator with respect to an orthonormal basis and vice-versa. There is a prominent place of Hankel operator in the study of operator theory. The study of Hankel operator was inspired by the initial progress of its first cousins (namely, Toeplitz operators). Hankel operators along with their matrix counterparts have appeared in several applications (namely, in realization problem related to discrete time linear systems and in determining exactly controllable systems [20, 22]. The operator equation $U^*X = XU$ characterizes Hankel operators on the Hardy space $H^2(\mathbb{D})$, where U is the unilateral forward shift operator on $H^2(\mathbb{D})$.

The first theorem for infinite Hankel matrices was given by Kronecker in the year 1881, which characterizes finite rank Hankel matrices. Nehari's work [14, 20, 21] shows that the Hankel operator can be derived by a function $\phi \in L^\infty(\mathbb{T})$. It is denoted by H_ϕ. Power [21] analyzed the essential spectrum of Hankel operators H_ϕ in terms of piecewise continuous functions ϕ in the space $L^\infty(\mathbb{T})$. A lot of progress has taken place in the study of Hankel operators on Bergman spaces on the disk, Dirichlet type spaces, Bergman and Hardy spaces on the unit ball in $\mathbb{C}^n$ or on symmetric domains, etc. [12, 21, 24]. For the fundamental terminologies and concepts related to Hankel operators, one is referred to [3, 20].

The development of theories of Toeplitz and Hankel operators along with their characterizations enlightened researchers to consider various modifications and manipulations of these equations, which give rise different generalizations of Toeplitz and Hankel operators

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like, slant Hankel operators, slant Toeplitz operators, λ-Hankel operators, λ- Toeplitz operators and (λ, μ)-Hankel operators etc (see [2, 7, 8, 9, 10, 11]). Few of them have already been investigated. For example, M. C. Ho [16, 15] introduced and analyzed the concept of slant Toeplitz operators on the space $L^2(\mathbb{T})$ along with their properties (basic as well as operator theoretic properties). The operator equation $SX = XT$ was considered by Ptak, where S and T are contractions. S. Power [21] analyzed and investigated simultaneous solutions of the system of operator equations $SX = XS$ for all operator S, which commutes with shift operators. In the year 2000, Avendano [4] discussed the operator equation $U^*X - XU = \lambda X$ with the motivation of work done by Barria and Halmos [5], where $\lambda \in \mathbb{C}$ and U is the unilateral forward shift operator on the Hardy space. Later in 2002, the work of Barria and Halmos [5] motivates Avendaño to initiate the notion of essentially Hankel operators, which is a generalization of Hankel operators in reference to the Calkin algebra.

A new variant of Hankel operators (namely, slant Hankel operators) was introduced in 2006 by S. C. Arora [1, 2] along with his research associates (motivated by the work of M.C. Ho [16]). The solutions of operator equation $M_z^* X = XM_{z^2}$ are known as slant Hankel operators [2]. A study of k^{th}-order slant Hankel operators, essentially slant Hankel operators and essentially λ-Hankel operators was done by Arora and Batra [1] on $L^2(\mathbb{T})$ and $H^2(\mathbb{T})$ repectively, which generalizes slant Hankel operators. The approach given by Avendaño [3] gave a thrust to consider another operator equation (namely, $\lambda M_z^* X = XM_{z^k}$ [9], for $k \geq 2$ and $\lambda \in \mathbb{C}$), which particularly resulted into the k^{th}-order slant Hankel operators for $\lambda = 1$.

It is well known that the space $L^2(\mathbb{T})$ is the Hilbert space consisting of all measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that the integral $\int_{\mathbb{T}} |f|^2 \, d\sigma$ is finite with respect to normalized Lebesgue measure $d\sigma$. Also, the Lebesgue space $L^\infty(\mathbb{T})$ contains all measurable functions $f : \mathbb{T} \to \mathbb{C}$, which are essentially bounded on $\mathbb{T}$. In terms of Fourier coefficients, the functions $\phi \in L^2(\mathbb{T})$ and $\psi \in H^2(\mathbb{T})$ can be written as

$$\phi(z) = \sum_{m\in\mathbb{Z}} \phi_m z^m, \quad \text{with} \quad \sum_{m\in\mathbb{Z}} |\phi_m|^2 < \infty,$$

and

$$\psi(z) = \sum_{m\in\mathbb{Z}_+} \psi_m z^m, \quad \text{with} \quad \sum_{m\in\mathbb{Z}_+} |\psi_m|^2 < \infty,$$

where $\mathbb{Z}$ and $\mathbb{Z}_+$ stands for the collections of all integers and non-negative integers respectively. It is well known that the function

$$\langle \phi, \psi \rangle = \frac{1}{(2\pi)} \int_0^{2\pi} \phi(e^{i\theta}) \overline{\psi(e^{i\theta})} d\theta,$$

defines an inner product on both the spaces $L^2(\mathbb{T})$ and $H^2(\mathbb{T})$, which makes them as a Hilbert space having orthonormal basis $\{e_m : m \in \mathbb{Z}\}$ and $\{e_m : m \in \mathbb{Z}_+\}$, respectively. The basis element e_m is given by $e_m(z) = z^m$. We frequently interchange z^m and e_m as a basis element, whenever there is no scope of confusion. Since the Hardy space $H^2(\mathbb{D})$ is identified with $H^2(\mathbb{T})$ in terms of the radial limits of functions of $H^2(\mathbb{D})$. Therefore, the space $H^2(\mathbb{D})$ can be considered as a closed subspace of $L^2(\mathbb{T})$, which consisting of all measurable functions $\phi : \mathbb{T} \to \mathbb{C}$ such that $\langle \phi, e_m \rangle = 0$, for all $m < 0$ (see [13]).

Datt and Aggarwal [9] have analyzed non-spectral problem (namely, $\lambda M_z^* X = XM_{z^2}$) whose solutions are given with the help of Hankel operator and composition operator. These solutions are another variants of λ- slant Hankel operators on the Lebesgue space $L^2(\mathbb{T})$ and are given by $T_{\phi,\lambda} = D_\lambda WJM_\phi$, where $D_\lambda(f(z)) = f(\bar{\lambda} z)$ (composition operator), $J(f(z)) = f(\bar{z})$ (flip operator), $M_\phi(f)(z) = \phi(z)f(z)$ (multiplication operator) and

$$W(z^n) = \begin{cases} z^{\frac{n}{2}}, & \text{if n is a multiple of 2} \\ 0, & \text{if n is not a multiple of 2.} \end{cases}$$

More recently, spectral problem (namely, $\mu M_z^* X - X M_{z^2} = \lambda X$) related to slant Hankel operator have been considered in [19, 18]. It gives rise new variants slant Hankel operators (namely, (λ, μ)- slant Hankel operators) on the space $L^2(\mathbb{T})$, which is a counterpart, in slanted version, of Hankel operator introduced by Avandaño [3].

In this paper, we take up the challenge of analyzing operators satisfying operator equations $(M_z^* - \lambda I)X - X M_{z^2} = K_1$ and $(\mu M_z^* - \lambda I)X - X M_{z^2} = K_2$ where $K_1, K_2 \in \mathcal{K}(L^2(\mathbb{T}))$. Here, $\mathcal{K}(L^2(\mathbb{T}))$ is an ideal of the C^*- algebra $\mathcal{B}(L^2(\mathbb{T}))$, which consists of all compact operators on $L^2(\mathbb{T})$. These problems give rise another variant of slant Hankel operators (namely, essentially λ- slant Hankel operators and essentially (λ, μ)- slant Hankel operators, respectively).

2. Essentially λ- Slant Hankel Operator

This section introduces an operator, which can be derived by an operator equation

$$(M_z^* - \lambda I)X - X M_{z^2} = 0, \quad mod(\mathcal{K}(L^2(\mathbb{T}))).$$

It also deals with certain elementary properties of these operators along with their relations with other class of operators. Initially, let us recall the definition of λ- Hankel operator and λ- slant Hankel operator for a given complex number $\lambda \in \mathbb{C}$.

Definition 1. *An operator Y on $H^2(\mathbb{T})$ is λ- Hankel operator* [3, 4] *if it satisfies operator equation $U^*Y - YU = \lambda Y$. In a similar way, an operator X, satisfying the operator equation $M_z^* X - X M_{z^2} = \lambda X$, is termed as λ- slant Hankel operator* [18, 19].

Now, we are in a position to introduce the notion of essentially λ- slant Hankel operator on the Lebesgue space $L^2(\mathbb{T})$.

Definition 2. *Let $\lambda \in \mathbb{C}$ be given. A bounded operator T on the Lebesgue space $L^2(\mathbb{T})$ such that the operator $(M_z^* - \lambda I)T - T M_{z^2} \in \mathcal{K}(L^2(\mathbb{T}))$ is known as essentially λ- slant Hankel operator.*

The collection of all essentially λ- slant Hankel operators on $L^2(\mathbb{T})$ is denoted by $esslantHank_\lambda(L^2)$. Since the characterization for slant Hankel operator X is given by $M_z^* X - X M_{z^2} = 0$. Therefore, X can also be seen as essentially λ- slant Hankel operator for $\lambda = 0$. In other words, the class $slantHank(L^2)$ (consisting of all slant Hankel operators) is contained in the class $esslantHank_\lambda(L^2)$.

From the definition of essentially λ- slant Hankel operator itself, the following can be easily derived.

(1) $\mathcal{K}(L^2(\mathbb{T})) \subsetneq esslantHank_\lambda(L^2)$.
(2) $slantHank_\lambda(L^2) \subsetneq esslantHank_\lambda(L^2)$, where $slantHank_\lambda(L^2)$ is the class of all λ- slant Hankel operators on $L^2(\mathbb{T})$.
(3) The identity operator I is not an essentially λ- slant Hankel operator, because of the fact that the multiplication operator M_ϕ is not compact operator on $L^2(\mathbb{T})$ for $\phi \neq 0$.

Now, we observe that the operator $X + K$ is a essentially λ- slant Hankel operator for a given $X \in esslantHank_\lambda(L^2)$ and $K \in \mathcal{K}(L^2(\mathbb{T}))$. In particular, every compact perturbation of λ- slant Hankel operator is also an essentially λ- slant Hankel operator. Next example evidently shows that each elements of $esslantHank_\lambda(L^2)$ not necessarily a compact perturbation of elements of $slantHank_\lambda(L^2)$.

Example 3. *There exists an operator $S \in esslantHank_\lambda(L^2)$ which is not a compact perturbation of any operator in $slantHank_\lambda(L^2)$.*

Proof. We know that the essentially slant Hankel operators and slant Hankel operators can be treated as operators of classes $esslantHank_\lambda(L^2)$ and $slantHank_\lambda(L^2)$ respectively for $\lambda = 0$. Therefore, here we produce an essentially slant Hankel operator, which can not be written as sum of slant Hankel operator and compact operator. For this, let us consider $\phi(e^{it}) = \text{sgn}(\sin t) \in L^\infty(\mathbb{T})$, which is discontinuous but not in VMO (Vanishing Mean Oscillation). Then the associated slant Hankel operator

$$T := S_\phi = WJM_\phi,$$

where the flip operator is given by $J(z^n) = z^{-n}$ and $W(z^n) = \begin{cases} z^{\frac{n}{2}}, & \text{if n is even} \\ 0, & \text{otherwise,} \end{cases}$
is not a compact operator since it has positive essential norm. It means that $T \in slantHank_0(L^2)$. But the coset of T in the Calkin algebra lies in $esslantHank_0(L^2)$.

Next, define an operator $S = T + D$, where D is the diagonal operator on L^2 given by

$$De_n = d_n e_n, \quad d_n = (-1)^{\lfloor \sqrt{n} \rfloor},$$

where $\lfloor ... \rfloor$ represents floor function. Since the diagonal sequence $\{d_n\}$ does not converge to 0 therefore the diagonal operator D is bounded but not compact. Thus S belongs to $esslantHank_0(L^2)$, as the cosets of S and T coincides in the Calkin algebra.

Suppose, for contradiction, that $S = S_\psi + K$ for some slant Hankel operator S_ψ and compact operator K. Then, we have

$$\|Se_n - S_\psi e_n\| = \|De_n + (T - S_\psi)e_n\|.$$

But $De_n = d_n e_n$ has norm 1 for infinitely many n, so the expression in the preceding equation does not converge to 0 as $n \to \infty$. We know that every compact operator maps weakly null sequences to norm-null sequences. Therefore, the operator $K = (S - S_\psi)$ cannot be compact, which is a contradiction. It implies that S cannot be written as a compact perturbation of any slant Hankel operator. Thus we have $S \in esslantHank_0(L^2)$ but $S \notin slantHank_0(L^2) + \mathcal{K}(L^2)$. □

It is well known that the class $esslantHank(L^2)$ of all essentially slant Hankel operators on $L^2(\mathbb{T})$ is not an algebra. Since $esslantHank_0(L^2) = esslantHank(L^2)$, therefore the collection $esslantHank_\lambda(L^2)$ need not be an algebra for $\lambda \in \mathbb{C}$. Now, we look for conditions under which the class $esslantHank_\lambda$ is closed.

Theorem 4. *The collection $esslantHank_\lambda(L^2)$ of all essentially λ- slant Hankel operators is a subspace of $\mathcal{B}(L^2(\mathbb{T}))$ and it is also norm closed in $\mathcal{B}(L^2(\mathbb{T}))$.*

Proof. Let T and S be any two operators of the space $esslantHank_\lambda(L^2)$ and a and b be any two complex numbers then the operators $(M_z^* - \lambda I)T - TM_{z^2}$ and $(M_z^* - \lambda I)S - SM_{z^2}$ are compact operators. Now, consider the expression

$$\begin{aligned} (M_z^* - \lambda I)(aT + bS) - (aT + bS)M_{z^2} &= a\{(M_z^* - \lambda I)T - TM_{z^2}\} \\ &\quad +b\{(M_z^* - \lambda I)S - SM_{z^2}\} \\ &= 0 \quad (mod\ \mathcal{K}(L^2(\mathbb{T}))) \end{aligned}$$

It gives that $(aT + bS) \in esslantHank_\lambda(L^2)$, hence $esslantHank_\lambda(L^2)$ is a subspace of $\mathcal{B}(L^2(\mathbb{T}))$.

For the next part, assume that (T_n) be a sequence of elements of $esslantHank_\lambda(L^2)$, which converges to T uniformly in $\mathcal{B}(L^2(\mathbb{T}))$, that is, $\|T_n - T\| \to 0$. Now, consider

$$\begin{aligned} &\|[(M_z^* - \lambda I)T_n - T_nM_{z^2}] - [(M_z^* - \lambda I)T - TM_{z^2}]\| \\ &\quad = \|\{(M_z^* - \lambda I)(T_n - T)\} - \{(T_n - T)M_{z^2}\}\| \\ &\quad \le \|(M_z^* - \lambda I)\|\|T_n - T\| + \|T_n - T\|\|M_{z^2}\| \\ &\quad \to 0. \end{aligned}$$

As we know that $\mathcal{K}(L^2(\mathbb{T}))$ is a closed ideal of $\mathcal{B}(L^2(\mathbb{T}))$ and each operator $(M_z^* - \lambda I)T_n - T_nM_{z^2}$ is compact for $n \in \mathbb{N}$, therefore above relation gives that the operator $(M_z^* - \lambda I)T - TM_{z^2}$ is also a compact operator. Hence, $T \in esslantHank_\lambda(L^2)$. □

The next theorem provides condition under which the space $esslantHank_\lambda(L^2)$ satisfies closure property with respect to composition of functions.

Theorem 5. *The product of two operators T and S of the space $esslantHank_\lambda(L^2)$ is again in the same space if and only if the operator $T[(M_z^* - \lambda I) - M_{z^2}]S$ is compact operator on $L^2(\mathbb{T})$.*

Proof. For given operators T and S of the space $esslantHank_\lambda(L^2)$, the expression $(M_z^* - \lambda I)TS - TSM_{z^2}$ can be written as

$$\begin{aligned}
(M_z^* - \lambda I)TS - TSM_{z^2} &= (M_z^* - \lambda I)TS - TM_{z^2}S + TM_{z^2}S - T(M_z^* - \lambda I)S \\
&\quad + T(M_z^* - \lambda I)S - TSM_{z^2} \\
&= [(M_z^* - \lambda I)T - TM_{z^2}]S + T[M_{z^2} - (M_z^* - \lambda I)]S \\
&\quad + T[(M_z^* - \lambda I)S - SM_{z^2}] \\
&= T[M_{z^2} - (M_z^* - \lambda I)]S \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))).
\end{aligned}$$

In the view of above relation, result follows trivially. □

As an immediate consequence of preceding theorem is given as follows.

Corollary 6. *For any essentially λ- slant Hankel operator T and positive integer n, the operator T^n is in the space $esslantHank_\lambda(L^2)$ if and only if either $T^p[(M_z^* - \lambda I) - M_{z^2}]T^q$ or $T^q[(M_z^* - \lambda I) - M_{z^2}]T^p$ is compact operator for some decomposition of $n = p + q$ such that $T^p, T^q \in esslantHank_\lambda(L^2)$.*

The following results give conditions under which the space $esslantHank_\lambda(L^2)$ is closed under the composition with the bounded operator.

Theorem 7. *Let T be an essentially λ- slant Hankel operator on $L^2(\mathbb{T})$. Then,*

1. *The operator $ST \in esslantHank_\lambda(L^2)$ for an essential commutant S of $(M_z^* - \lambda I)$.*
2. *The operator $TS \in esslantHank_\lambda(L^2)$ for an essential commutant S of M_{z^2}.*

Proof. Let $T \in esslantHank_\lambda(L^2)$ and S be an essential commutant of $(M_z^* - \lambda I)$. Then, the expression $(M_z^* - \lambda I)ST - STM_{z^2}$ reduces to

$$\begin{aligned}
(M_z^* - \lambda I)ST - STM_{z^2} &= S(M_z^* - \lambda I)T - STM_{z^2} \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))) \\
&= S[(M_z^* - \lambda I)T - TM_{z^2}] \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))) \\
&= 0, \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))).
\end{aligned}$$

Similarly, consider an operator $(M_z^* - \lambda I)TS - TSM_{z^2}$ for an essential commutant S of M_{z^2}, which can be re-written as

$$\begin{aligned}
(M_z^* - \lambda I)TS - TSM_{z^2} &= (M_z^* - \lambda I)TS - TM_{z^2}S \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))) \\
&= [(M_z^* - \lambda I)T - TM_{z^2}]S \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))) \\
&= 0, \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))).
\end{aligned}$$

This proves the desired result trivially. □

We know that the multiplication operator M_ϕ for some $\phi \in L^\infty(\mathbb{T})$ is a commutant of both the operators $(M_z^* - \lambda I)$ and M_{z^2}. Therefore, it is also an essential commutant of $(M_z^* - \lambda I)$ as well as M_{z^2}. Hence, one can derive the following result as a particular case of preceding theorem. But, here, we also provide an independent proof of this one.

Corollary 8. *The space* $esslantHank_{\lambda}(L^2)$ *is closed under the product of multiplication operator* M_{ϕ} *on* $L^2(\mathbb{T})$ *for* $\phi \in L^{\infty}(\mathbb{T})$.

Proof. Assume that T is an arbitrary element of the class $esslantHank_{\lambda}(L^2)$ and $\phi \in L^{\infty}(\mathbb{T})$. Then

$$\begin{aligned} (M_z^* - \lambda I)TM_{\phi} - TM_{\phi}M_{z^2} &= [(M_z^* - \lambda I)T - TM_{z^2}]M_{\phi} \\ &= 0, \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} (M_z^* - \lambda I)M_{\phi}T - M_{\phi}TM_{z^2} &= M_{\phi}[(M_z^* - \lambda I)T - TM_{z^2}] \\ &= 0, \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))). \end{aligned} \tag{2.2}$$

The equations (2.1) and (2.2) give that the operators TM_{ϕ} and $M_{\phi}T$ are essentially λ-slant Hankel operators. □

Theorem 9. *If an operator* T *of the space* $esslantHank_{\lambda}(L^2)$ *is skew self-adjoint (or skew-Hermitian). Then* TM *is always essentially self-adjoint for* $M = (M_z - \bar{\lambda}I) - M_{z^2}$.

Proof. Let T be an essentially λ- slant Hankel operator on $L^2(\mathbb{T})$, which is skew self-adjoint, then T as well as T^* belongs to $esslantHank_{\lambda}(L^2)$. Therefore, we have $(M_z^* - \lambda I)T - TM_{z^2} = K_1$ and $(M_z^* - \lambda I)T^* - T^*M_{z^2} = K_2$ for some compact operators K_1 and K_2. From the first expression, we obtain $T^*(M_z - \bar{\lambda}I) - M_{z^2}^*T^* = K_1^*$. It yields that

$$-T^*\{(M_z - \bar{\lambda}I) - M_{z^2}\} = \{(M_z^* - \lambda I) - M_{z^2}^*\}T^* \quad (mod\mathcal{K}(L^2(\mathbb{T}))). \tag{2.3}$$

Since T is skew self-adjoint, that is, $T^* = -T$ therefore we have

$$T\{(M_z - \bar{\lambda}I) - M_{z^2}\} = \{(M_z^* - \lambda I) - M_{z^2}^*\}T^* \quad (mod\mathcal{K}(L^2(\mathbb{T}))), \tag{2.4}$$

which trivially implies that the operator TM is essentially self-adjoint, where $M = \{(M_z - \bar{\lambda}I) - M_{z^2}\}$. □

3. Essentially (λ, μ)- Slant Hankel Operator

This section starts with similar question as asked by Barria and Halmos [5], (namely, "What are the operators satisfying the operator equation $U^*XU = \lambda X$ for $\lambda \in \mathbb{C}$?") in Calkin algebra. Here, we investigate (λ, μ)- slant Hankel operators in the Calkin algebra. More precisely, we look for an operator Y such that the operator $(\mu M_z^* - \lambda I)Y - YM_{z^2}$ is compact operator on $L^2(\mathbb{T})$. In this case, the operator Y is known as essentially (λ, μ)-slant Hankel operators. The (λ, μ)- slant Hankel operators are obvious choices for the operator, which are being looked for. If we choose X to be a compact operator then the operator $(\mu M_z^* - \lambda I)X - XM_{z^2}$ is also compact operator on $L^2(\mathbb{T})$. Thus, we can say that every compact operator is essentially (λ, μ)- slant Hankel operator. It is happening because of the fact that every compact operator can be considered as zero operator in the Calkin algebra. Now, consider a problem of searching an operator X such that

$$\mu M_z^* X - XM_{z^2} - \lambda X \in \mathcal{K}(L^2(\mathbb{T})), \tag{3.5}$$

which is similar to spectral type problem in the Calkin algebra, which leads to the study of another variant of slant Hankel operators (namely, essentially (λ, μ)- slant Hankel operators). These operators can be seen as upgradation of λ-Hankel and slant Hankel operators. Here, we also investigate several properties of essentially (λ, μ)- slant Hankel operators, including basic as well as alegebraic properties and spectral properties. Next, we list certain known results, terminologies, and definitions, which are useful during our analysis.

Definition 10 (Slant Hankel Operator)**.** [1, 2] *An operator X on $L^2(\mathbb{T})$ satisfying the equation $M_z^* X = X M_{z^2}$ is known as slant Hankel operator on the space $L^2(\mathbb{T})$. In terms of multiplication operator, it is given by $A_\phi = WJM_\phi$, where J is the flip operator, M_ϕ is multiplication operator on $L^2(\mathbb{T})$ for some $\phi \in L^\infty(\mathbb{T})$ and* $W(z^n) = \begin{cases} z^{\frac{n}{2}} & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd.} \end{cases}$

Next definitions of essential Hankel operator and (λ, μ)- Hankel operator are due to Avendaño [3, 4] and Datt, Aggarwal [8] respectively.

Definition 11 (Essential Hankel Operator)**.** [3, 4] *An operator X on the Hardy space $H^2(\mathbb{T})$ such that the operator $(U^* - \lambda I)X - XU \in \mathcal{K}(H^2(\mathbb{T}))$ is known as essential Hankel operator, where U is a unilateral shift operator on $H^2(\mathbb{T})$.*

Definition 12 ((λ, μ)- Hankel Operator)**.** [7, 8] *An operator X on the Hardy space $H^2(\mathbb{T})$ satisfying the operator equation $\mu U^* X - XU = \lambda X$ is known as (λ, μ)- Hankel operator, where U is again forward shift operator. In a similar way, an operator X on the Lebesgue space $L^2(\mathbb{T})$, satisfying operator equation $\mu M_z^* X - X M_{z^2} = \lambda X$, is called (λ, μ)- slant Hankel operator* [18, 19]*. The collection of all (λ, μ)- slant Hankel operators is denoted by $slantHank_{(\lambda,\mu)}(L^2)$.*

Next, we are going to provide the concept of (λ, μ)- slant Hankel operator on $L^2(\mathbb{T})$ in the Calkin algebra $\frac{\mathcal{B}(L^2)}{\mathcal{K}(L^2)}$.

Definition 13. *For given complex numbers λ and μ, bounded operators T on $L^2(\mathbb{T})$, characterized by the operator equation $(\mu M_z^* - \lambda I)T - TM_{z^2} = K$ for some $K \in \mathcal{K}(L^2(\mathbb{T}))$, are known as essentially (λ, μ)- slant Hankel operators.*

The class $esslantHank_{\lambda,\mu}(L^2)$ is the set of all essentially (λ, μ)- slant Hankel operators on $L^2(\mathbb{T})$. Let X be a slant Hankel operator, then it can be characterize by the operator equation $M_z^* X - X M_{z^2} = 0$. Thus, we can consider X to be an essentially (λ, μ)- slant Hankel operator for $\lambda = 0$ and $\mu = 1$. Therefore, the class $slantHank(L^2)$ (containing slant Hankel operators) is a subset of the space $esslantHank_{(\lambda,\mu)}(L^2)$.

By the definition, it is easy to observe that each (λ, μ)- slant Hankel operator is also an essentially (λ, μ)- slant Hankel operator. Let $T \in esslantHank_{(\lambda,\mu)}(L^2)$, then consider the operator $(\mu M_z^* - \lambda I)(T + K) - (T + K)M_{z^2}$, which turns out to be a compact operator for each compact operator K on $L^2(\mathbb{T})$. In other words, we can say that every compact perturbation of essentially (λ, μ)- slant Hankel operator (particularly, slant Hankel operator, (λ, μ)- slant Hankel operator) is again essentially (λ, μ)- slant Hankel operator. Since, for $\lambda = 0$ and $\mu = 1$, essentially (λ, μ)- slant Hankel operator reduces to essentially slant Hankel operator. Therefore in the view of Example 3, we can conclude that every element of $esslantHank_{(\lambda,\mu)}(L^2)$ need not be a compact perturbation of (λ, μ)- slant Hankel operator. We know that the space $esslantHank(L^2)$ (containing essentially slant Hankel operators) is not an algebra. Since $esslantHank_{0,1}(L^2) = esslantHank(L^2)$, therefore it implies that the class $esslantHank_{\lambda,\mu}(L^2)$ is not necessarily an algebra for complex numbers λ and μ. The following result provides the closedness condition for the class $esslantHank_{(\lambda,\mu)}(L^2)$.

Theorem 14. *The class $esslantHank_{(\lambda,\mu)}(L^2)$ is closed under the composition if and only if the condition $X[M_{z^2} - (\mu M_z^* - \lambda I)]Y \in \mathcal{K}(L^2(\mathbb{T}))$ is satisfied for all operators X and Y of the class $esslantHank_{(\lambda,\mu)}(L^2)$. Here, XY represents the composition of X and Y.*

Proof. For the given operators X and Y from the class $esslantHank_{(\lambda,\mu)}(L^2)$, we obtain the relations $(\mu M_z^* - \lambda I)X - X M_{z^2} \in \mathcal{K}(L^2(\mathbb{T}))$ and $(\mu M_z^* - \lambda I)Y - Y M_{z^2} \in \mathcal{K}(L^2(\mathbb{T}))$.

Utilizing it, we get that

$$\begin{aligned}(\mu M_z^* - \lambda I)XY - XYM_{z^2} &= (\mu M_z^* - \lambda I)XY - XM_{z^2}Y + XM_{z^2}Y \\ &\quad -X(\mu M_z^* - \lambda I)Y + X(\mu M_z^* - \lambda I)Y - XYM_{z^2} \\ &= \{(\mu M_z^* - \lambda I)X - XM_{z^2}\}Y + X\{M_{z^2} - (\mu M_z^* - \lambda I)\}Y \\ &\quad +X\{(\mu M_z^* - \lambda I)Y - YM_{z^2}\} \\ &= X\{M_{z^2} - (\mu M_z^* - \lambda I)\}Y \quad (mod\, \mathcal{K}(L^2(\mathbb{T}))) \end{aligned} \tag{3.6}$$

The preceding relation (3.6) trivially yields that $XY \in esslantHank_{(\lambda,\mu)}(L^2)$ if and only if the operator $X[M_{z^2} - (\mu M_z^* - \lambda I)]Y$ is a compact operator. This completes the proof. □

The following result is an immediate consequence of Theorem (14), which can be written as follows.

Corollary 15. *For an operator $T \in esslantHank_{(\lambda,\mu)}(L^2)$ and $n \in \mathbb{N}$, the operator $T^n \in esslantHank_{(\lambda,\mu)}(L^2)$ if and only if either $T^p[(\mu M_z^* - \lambda I) - M_{z^2}]T^q$ or $T^q[(\mu M_z^* - \lambda I) - M_{z^2}]T^p$ is an element of $\mathcal{K}(L^2(\mathbb{T}))$ for a decomposition of n in terms of p and q such that $T^p, T^q \in esslantHank_{(\lambda,\mu)}(L^2)$.*

The next theorem depicts the uniform closedness of $esslantHank_{(\lambda,\mu)}(L^2)$ in the space $\mathcal{B}(L^2(\mathbb{T}))$ of all bounded operators on $L^2(\mathbb{T})$.

Theorem 16. *The class $esslantHank_{(\lambda,\mu)}(L^2)$ is a norm closed subspace of $\mathcal{B}(L^2(\mathbb{T}))$.*

Proof. Let T and S belong to the class $esslantHank_{(\lambda,\mu)}(L^2)$ then we get two compact operators namely, $(\mu M_z^* - \lambda I)T - TM_{z^2}$ and $(\mu M_z^* - \lambda I)S - SM_{z^2}$. For given complex numbers a and b, we obtain that

$$\begin{aligned}(\mu M_z^* - \lambda I)(aT + bS) - (aT + bS)M_{z^2} &= a\{(\mu M_z^* - \lambda I)T - TM_{z^2}\} \\ &\quad +b\{(\mu M_z^* - \lambda I)S - SM_{z^2}\}\end{aligned}$$

From the above observation, it is easy to derive that $(aT + bS) \in esslantHank_{(\lambda,\mu)}(L^2)$. Therefore the class $esslantHank_{(\lambda,\mu)}(L^2)$ of all essentially (λ,μ)- slant Hankel operators is a subspace of $\mathcal{B}(L^2(\mathbb{T}))$.
Next suppose that the operator S is the uniform operator limit of the sequence (S_n) of operators from the space $esslantHank_{(\lambda,\mu)}(L^2)$ in $\mathcal{B}(L^2(\mathbb{T}))$. It yields that

$$\begin{aligned}&\|[(\mu M_z^* - \lambda I)S_n - S_nM_{z^2}] - [(\mu M_z^* - \lambda I)S - SM_{z^2}]\| \\ &\quad = \|\{(\mu M_z^* - \lambda I)(S_n - S)\} - \{(S_n - S)M_{z^2}\}\| \\ &\quad \le \|(\mu M_z^* - \lambda I)\|\|S_n - S\| + \|S_n - S\|\|M_{z^2}\|\end{aligned}$$

The preceding observation yields that the operator $(\mu M_z^* - \lambda I)S_n - S_nM_{z^2}$ converges uniformly to the operator $(\mu M_z^* - \lambda I)S - SM_{z^2}$. Since $\mathcal{K}(L^2(\mathbb{T}))$ is a closed with respect to norm topology in $\mathcal{B}(L^2(\mathbb{T}))$ and each operator $(\mu M_z^* - \lambda I)S_n - S_nM_{z^2}$ is compact for each $n \in \mathbb{N}$. Therefore the operator $(M_z^* - \lambda I)S - SM_{z^2} \in \mathcal{K}(L^2(\mathbb{T}))$, which implies that $S \in esslantHank_{(\lambda,\mu)}(L^2)$. Hence, the result follows. □

The following theorem yields the condition, which is required to show the closedness of the space $esslantHank_{(\lambda,\mu)}(L^2)$ under composition.

Theorem 17. *For an operator $T \in esslantHank_{(\lambda,\mu)}(L^2)$, the operators ST and TR are again essentially (λ,μ)- slant Hankel operators, where S and R are essential commutants of $(\mu M_z^* - \lambda I)$ and M_{z^2} respectively.*

Proof. Let S be an essential commutant of $(\mu M_z^* - \lambda I)$. Then, the expression $(\mu M_z^* - \lambda I)ST - STM_{z^2}$ can be written as

$$\begin{aligned}(\mu M_z^* - \lambda I)ST - STM_{z^2} &= S(\mu M_z^* - \lambda I)T - STM_{z^2} + K_1 \\ &= S[(\mu M_z^* - \lambda I)T - TM_{z^2}] + K_1,\end{aligned}$$

for some $K_1 \in \mathcal{K}(L^2(\mathbb{T}))$. It reduces to zero operator in the Calkin algebra as $T \in esslantHank_{(\lambda,\mu)}(L^2)$.
In a similar way, for an essential commutant R of M_{z^2}, the expression $(\mu M_z^* - \lambda I)TR - TRM_{z^2}$ reduces to

$$\begin{aligned}(\mu M_z^* - \lambda I)TR - TRM_{z^2} &= (\mu M_z^* - \lambda I)TR - TM_{z^2}R + K_2 \\ &= [(\mu M_z^* - \lambda I)T - TM_{z^2}]R + K_2,\end{aligned}$$

for some compact operator K_2 on $L^2(\mathbb{T})$. It implies that the operator $(\mu M_z^* - \lambda I)TR - TRM_{z^2}$ is a compact operator and hence $TR \in esslantHank_{(\lambda,\mu)}(L^2)$. This completes the theorem. □

The following outcome shows that the class $esslantHank_{(\lambda,\mu)}(L^2)$ is also closed under the composition of multiplication operator M on $L^2(\mathbb{T})$.

Theorem 18. *Let M be a multiplication operator. Then the operators TM and MT are elements of the space* $esslantHank_{(\lambda,\mu)}(L^2)$ *for any* $T \in esslantHank_{(\lambda,\mu)}(L^2)$.

Proof. Let M be a multiplication operator on $L^2(\mathbb{T})$ and T be an essentially (λ, μ)- slant Hankel operator. Then the expression $(\mu M_z^* - \lambda I)(TM) - (TM)M_{z^2}$ reduces to

$$(\mu M_z^* - \lambda I)(TM) - (TM)M_{z^2} = [(\mu M_z^* - \lambda I)T - TM_{z^2}]M. \tag{3.7}$$

Similarly, we have

$$(\mu M_z^* - \lambda I)(MT) - (MT)M_{z^2} = M[(\mu M_z^* - \lambda I)T - TM_{z^2}]. \tag{3.8}$$

Utilizing the fact that $T \in esslantHank_{(\lambda,\mu)}(L^2)$, the relations (3.7) and (3.8) help to conclude that TM and MT are also in the same space $esslantHank_{(\lambda,\mu)}(L^2)$. □

Note that the multiplication operator M lies in the essential commutant of both the operators $(\mu M_z^* - \lambda I)$ and M_{z^2}. Therefore, the above theorem can also be retrieved as a particular case of the Theorem 17. The following result is important in the sense that it provides self adjoint operator in the Calkin algebra in terms of skew self adjoint operator from the space $esslantHank_{(\lambda,\mu)}(L^2)$.

Theorem 19. *The operator TS is Hermitian in the Calkin algebra for a given skew-Hermitian self-adjoint operator* $T \in esslantHank_{(\lambda,\mu)}(L^2)$, *where* $S = \{(\bar{\mu}M_z - \bar{\lambda}I) - M_{z^2}\}$.

Proof. Let T be a skew-Hermitian operator in the space $esslantHank_{(\lambda,\mu)}(L^2)$ then $T, T^* \in esslantHank_{(\lambda,\mu)}(L^2)$. It implies that

$$\begin{aligned}(\mu M_z^* - \lambda I)T - TM_{z^2} &= K_1 \\ T^*(\bar{\mu}M_z - \bar{\lambda}I) - M_{z^2}^*T^* &= K_1^*,\end{aligned} \tag{3.9}$$

and

$$(\mu M_z^* - \lambda I)T^* - T^*M_{z^2} = K_2, \tag{3.10}$$

where operators K_1 and K_2 are compact. Adding relations (3.9) and (3.10), we obtain that

$$-T^*\{(\bar{\mu}M_z - \bar{\lambda}I) - M_{z^2}\} = \{(\mu M_z^* - \lambda I) - M_{z^2}^*\}T^* \quad (mod\mathcal{K}(L^2(\mathbb{T}))).$$

According to assumption, T is a skew self-adjoint operator, therefore $T^* = -T$, which gives that

$$T\{(\bar{\mu}M_z - \bar{\lambda}I) - M_{z^2}\} = \{(\mu M_z^* - \lambda I) - M_{z^2}^*\}T^* \quad (mod\mathcal{K}(L^2(\mathbb{T}))).$$

From the above expression, it is obvious that TS is essentially Hermitian operator, where $S = \{(\bar{\mu}M_z - \bar{\lambda}I) - M_{z^2}\}$. □

4. Connection between Two Spaces $esslantHank_{(\lambda)}(L^2)$ and $esslantHank_{(\lambda,\mu)}(L^2)$

In this section, we shall establish certain results, which depicts the connecting bridge between spaces $esslantHank_{(\lambda)}(L^2)$ and $esslantHank_{(\lambda,\mu)}(L^2)$. Let μ be a fixed complex number. Then, the operator $B_\mu : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ given by $B_\mu(e_n) = \mu^n e_n$ is a bounded linear operator provided that $|\mu| \leq 1$. A simple computation furnishes the following relations:

$$B_\mu M_z^*(e_n) = \frac{1}{\mu} M_z^* B_\mu(e_n),$$

and

$$B_\mu M_{z^2}(e_n) = \mu^{n+2} e_{n+2} = \mu^2 M_{z^2} B_\mu(e_n).$$

This implies $B_\mu M_z^* = \frac{1}{\mu} M_z^* B_\mu$ and $B_\mu M_{z^2} = \mu^2 M_{z^2} B_\mu$. With the help of these relation, one can deduce following result, which connects essentially λ- slant Hankel operators and essentially (λ, μ)- slant Hankel operators.

Theorem 20. *Let μ be a complex number such that $|\mu| \leq 1$. Then, the following results are true.*

(1) *If $T \in esslantHank_{(\lambda,\mu)}(L^2)$, then the operator $B_\mu T \in esslantHank_\lambda(L^2)$.*
(2) *If $T \in esslantHank_\lambda(L^2)$, then the operator $TB_\mu \in esslantHank_{(\mu^2\lambda,\mu^2)}(L^2)$.*

Proof. Let $T \in esslantHank_{(\lambda,\mu)}(L^2)$. Then, we get $(\mu M_z^* - \lambda I)T - TM_{z^2} = 0$ in the Calkin algebra. It again reduces to $(\mu B_\mu M_z^* - \lambda B_\mu)T - B_\mu T M_{z^2} = 0$ in the Calkin algebra by pre-multiplying with B_μ. The connection between B_μ and M_z^*, observed in the above paragraph, yields that

$$(M_z^* - \lambda I)(B_\mu T) - (B_\mu T)M_{z^2} = 0 \qquad mod(\mathcal{K}(L^2(\mathbb{T}))).$$

Hence, it implies that $B_\mu T \in esslantHank_\lambda(L^2)$.
The second part starts with the assumption that $T \in esslantHank_\lambda(L^2)$. Then, it reduces to

$$(M_z^* - \lambda I)T - TM_{z^2} = 0 \qquad mod(\mathcal{K}(L^2(\mathbb{T}))).$$

On post-multiplying the above expression by B_μ, we have

$$(M_z^* - \lambda I)TB_\mu - TM_{z^2}B_\mu = 0 \qquad mod(\mathcal{K}(L^2(\mathbb{T}))).$$

Utilizing the connecting relation between B_μ and M_{z^2}, we obtain that the operator $(\mu^2 M_z^* - \mu^2\lambda I)(TB_\mu) - (TB_\mu)M_{z^2}$ is the zero operator in the Calkin algebra. It trivially yields that $TB_\mu \in esslantHank_{(\mu^2\lambda,\mu^2)}(L^2)$. Thus the proof follows. □

In the same lines as above, one may have following result for complex number μ such that $|\mu| \geq 1$.

Theorem 21. *Let $\mu \in \mathbb{C}$ such that $|\mu| \geq 1$. Then, the following results can be obtained.*

(1) *If $T \in esslantHank_{(\lambda,\mu^2)}(L^2)$, then the operator $TB_{\frac{1}{\mu}} \in esslantHank_{\frac{\lambda}{\mu^2}}(L^2)$.*
(2) *If $T \in esslantHank_\lambda(L^2)$, then the operator $B_{\frac{1}{\mu}}T \in esslantHank_{(\lambda,\mu)}(L^2)$.*

Proof. Assume that $T \in esslantHank_{(\lambda,\mu^2)}(L^2)$. Then, the operator $(\mu^2 M_z^* - \lambda I)T - TM_{z^2}$ is the zero operator in the Calkin algebra. On post-multiplying with $B_{\frac{1}{\mu}}$, the preceding operator again reduces to an operator $(\mu^2 M_z^* - \lambda I)TB_{\frac{1}{\mu}} - TM_{z^2}B_{\frac{1}{\mu}}$, which is the zero operator in the Calkin algebra. In fact, it implies that the operator $(\mu^2 M_z^* - \lambda I)(TB_{\frac{1}{\mu}}) - \mu^2(TB_{\frac{1}{\mu}})M_{z^2}$ is compact operator on $L^2(\mathbb{T})$ in the view of relation between $B_{\frac{1}{\mu}}$ and M_{z^2}. Thus, we conclude that the operator $TB_{\frac{1}{\mu}} \in esslantHank_{\frac{\lambda}{\mu^2}}(L^2)$. Further assume that T is a essentially λ- slant Hankel operator on $L^2(\mathbb{T})$, then $(M_z^* - \lambda I)T - TM_{z^2} = 0$ in the Calkin algebra. With the fact $B_{\frac{1}{\mu}} M_z^* = \mu M_z^* B_{\frac{1}{\mu}}$, we obtain that

$$B_{\frac{1}{\mu}}(M_z^* - \lambda I)T - B_{\frac{1}{\mu}}TM_{z^2} = 0, \quad \text{that is,} \quad (\mu M_z^* - \lambda I)(B_{\frac{1}{\mu}}T) - (B_{\frac{1}{\mu}}T)M_{z^2} = 0,$$

in the Calkin algebra on pre-multiplying by $B_{\frac{1}{\mu}}$. Hence, we can conclude that the operator $B_{\frac{1}{\mu}}T$ is in the space $esslantHank_{(\lambda,\mu)}(L^2)$. This completes the proof. □

The following corollary can be easily derived from the preceding result in terms of essentially (λ, μ)- slant Hankel operator.

Corollary 22. *Let μ be complex number with $|\mu| \geq 1$ and T be an essentially (λ, μ)- slant Hankel operator. Then, the operator $TB_{\frac{1}{\sqrt{\mu}}} \in esslantHank_{\frac{\lambda}{\mu}}(L^2)$.*

As we know that there is a way to obtain essentially λ- slant Hankel operator by assigning particular value "$\mu = 1$" in the essentially (λ, μ)- slant Hankel operator on $L^2(\mathbb{T})$. But the preceding Theorems 20, 21 and Corollary 22, provide another way to get the elements from the space $esslantHank_\lambda(L^2)$ in terms of elements of another space $esslantHank_{(\lambda,\mu)}(L^2)$ and vice-versa with the help of suitably chosen value of μ in the operator B_μ.

Note that the corollary 2.18 of [18] yields that there are infinite number of elements of the class $slantHank_{(\lambda,\mu)}(L^2)$ for $|\lambda| < 2$. Therefore, we can conclude that there are infinitely many elements of the spaces $esslantHank_{(\lambda,\mu)}(L^2)$ as well as $esslantHank_\lambda(L^2)$.

The following theorem depicts the relationship amongst different operators from the space $esslantHank_\lambda(L^2)$ corresponding to different choices of uni-modular complex numbers λ.

Theorem 23. *Let λ and δ be two uni-modular complex numbers. If T be an essentially (λ, μ)- slant Hankel operator. Then, the operator S given by $S = B_{\frac{1}{\delta\lambda}}TB_{\frac{1}{\sqrt{\delta\lambda}}}$ is an operator of the class $esslantHank_\delta(L^2)$, where $B_a(e_n) = a^n e_n$, for all $n \in \mathbb{Z}$, is a unitary operator.*

Proof. We know that the operators B_a and M_ϕ satisfy the relations $B_{\frac{1}{a}}M_z^* = aM_z^*B_{\frac{1}{a}}$ and $M_{z^2}B_{\frac{1}{a}} = a^2 B_{\frac{1}{a}}M_{z^2}$. Let $T \in esslantHank_\lambda(L^2)$, then the operator $(M_z^* - \lambda I)T - TM_{z^2}$ is the zero operator in the Calkin algebra. On considering the operator $S = B_{\frac{1}{\delta\lambda}}TB_{\frac{1}{\sqrt{\delta\lambda}}}$, the above relations between B_a and M_ϕ yield that

$$\begin{aligned}
B_{\frac{1}{\delta\lambda}}(M_z^* - \lambda I)TB_{\frac{1}{\sqrt{\delta\lambda}}} - B_{\frac{1}{\delta\lambda}}(TM_{z^2})B_{\frac{1}{\sqrt{\delta\lambda}}} &= 0 \quad mod(\mathcal{K}(L^2(\mathbb{T}))) \\
(\bar{\delta}\lambda M_z^* B_{\frac{1}{\delta\lambda}} - \lambda B_{\frac{1}{\delta\lambda}})TB_{\frac{1}{\sqrt{\delta\lambda}}} - \bar{\delta}\lambda B_{\frac{1}{\delta\lambda}}TB_{\frac{1}{\sqrt{\delta\lambda}}}M_{z^2} &= 0 \quad mod(\mathcal{K}(L^2(\mathbb{T}))) \\
(M_z^* - \delta I)(B_{\frac{1}{\delta\lambda}}TB_{\frac{1}{\sqrt{\delta\lambda}}}) - (B_{\frac{1}{\delta\lambda}}TB_{\frac{1}{\sqrt{\delta\lambda}}})M_{z^2} &= 0 \quad mod\left(\mathcal{K}(L^2(\mathbb{T}))\right).
\end{aligned}$$

It results into the fact that the operator $(M_z^* - \delta I)S - SM_{z^2}$ is the zero operator in the Calkin algebra. Hence we conclude that the operator $S = \left(B_{\frac{1}{\delta\lambda}}TB_{\frac{1}{\sqrt{\delta\lambda}}}\right)$ is a member of the space $esslantHank_\delta(L^2)$. □

Using the same technique, similar kind of result, as the preceding one, can be derived for the elements of the space $esslantHank_{\lambda,\mu}(L^2)$, which shows how the different operators of $esslantHank_{\lambda,\mu}(L^2)$ are linked to each other corresponding to different pairs of unimodular complex numbers λ, μ.

Theorem 24. *Let (α, β), and (λ, μ) be two pairs of uni-modular complex numbers and $T \in esslantHank_{\lambda,\mu}(L^2)$. Then, the operator $S = B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}$ is an element of the space $esslantHank_{\alpha,\beta}(L^2)$, where B_a is the same operator as defined in the preceding theorem.*

Proof. Since $T \in esslantHank_{\lambda,\mu}(L^2)$, therefore the operator $(\mu M_z^* - \lambda I)T - TM_{z^2} = 0$ in the Calkin algebra. Now consider the operator

$$B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} \left((\mu M_z^* - \lambda I)T - TM_{z^2}\right) B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}.$$

The relation between B_a and multiplication operator M_ϕ forces the above operator to have the following form in the Calkin algebra.

$$\begin{aligned} B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} \left((\mu M_z^* - \lambda I)T - TM_{z^2}\right) B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} &= 0 \\ \left(\mu B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} M_z^* - \lambda B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}}\right) T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} - B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} T M_{z^2} B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} &= 0 \\ \left(\mu\bar{\alpha}\beta\lambda\bar{\mu} M_z^* B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} - \lambda B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}}\right) T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} - \bar{\alpha}\lambda B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} M_{z^2} &= 0 \\ (\bar{\alpha}\beta\lambda M_z^* - \lambda I)\left(B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}\right) - \bar{\alpha}\lambda \left(B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}\right) M_{z^2} &= 0. \end{aligned} \tag{4.11}$$

On multiplying the equation (4.11) by a complex number $\alpha\bar{\lambda}$, we get that the operator $(\beta M_z^* - \alpha I)\left(B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}\right) - \left(B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}\right) M_{z^2}$ is compact operator on $L^2(\mathbb{T})$. Now, it trivially implies that $S = \left(B_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} T B_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}\right) \in esslantHank_{\alpha,\beta}(L^2)$. ☐

Next, we investigate invertibility in general as well as in Calkin algebra of elements of spaces $esslantHank_{\lambda,\mu}(L^2)$ and $esslantHank_{\lambda}(L^2)$.

Theorem 25. *Let T and S be two operators from the spaces $esslantHank_{\lambda,\mu}(L^2)$ and $esslantHank_{\lambda}(L^2)$, respectively. Then, both the operators T and S are not invertible as well as not essentially invertible operators, that is, $0 \in \sigma(T) \cap \sigma(S) \cap \sigma(T)_\epsilon \cap \sigma_\epsilon(S)$, where $\sigma(X)$ and $\sigma_\epsilon(X)$ represent the spectrum and the essential spectrum of X respectively.*

Proof. Let $T \in esslantHank_{\lambda,\mu}(L^2)$ and $S \in esslantHank_{\lambda}(L^2)$. If operators T and S are the zero operator, then there is nothing to prove. Next, assume that both the operators T and S are non-zero as well as invertible. Then, in Calkin algebra,

$$(\mu M_z^* - \lambda I)T - TM_{z^2} = 0 \quad \text{and} \quad (M_z^* - \lambda I)S - SM_{z^2} = 0. \tag{4.12}$$

The invertibility of operators T and S results into the fact that M_{z^2} as well as $(M_z^* - \lambda I)$ are essentially similar operators, which is incorrect. Therefore, our assumption is wrong, and hence T and S are non-invertible operator. It implies that $0 \in \sigma(T) \cap \sigma(S)$.
For the next part, let $T \in esslantHank_{\lambda,\mu}(L^2)$ and $S \in esslantHank_{\lambda}(L^2)$, which are non-zero and essentially invertible. Similarly, in the view of relation (4.12) in Calkin algebra and essentially invertibility of operators T and S, we arrive at the same conclusion (namely, M_{z^2} and $(\mu M_z^* - \lambda I)$ are essentially similar). It is again not true. Therefore, operators T and S are essentially non-invertible and hence $0 \in \sigma_\epsilon(T) \cap \sigma_\epsilon(S)$. Merging the outcomes of the above discussion, it is easy to point out that $0 \in \sigma(T) \cap \sigma(S) \cap \sigma(T)_\epsilon \cap \sigma_\epsilon(S)$. Thus, the proof is completed. ☐

5. Conclusion

In this paper, we have analyzed and investigated essentially λ- slant Hankel operators and essentially (λ, μ)- slant Hankel operators along their elementary properties and certain connections between themselves. Our study in this paper not merely includes results appearing in this paper but also contain many results (as a particular case of our study) of several eminent mathematicians, working in this area. Therefore, the impact and effect of our study is not only confined to this paper but also have impact and influence on previously well established results.

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