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SPLICED DOUBLE SEQUENCES AND DOUBLE DENSITY OF POINTS

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Abstract. In this manuscript we define A-double density of a given point in $\mathbb{R}$ for double sequences. We state the relation between double density of a point and spliced double sequences and also examine limit points in view of density of a point.

1. Preliminaries

Osikiewicz [5] first introduced the notion of spliced sequences, aiming to provide an alternative proof demonstrating the existence of bounded sequences that cannot be summed by a regular matrix. Since then, spliced sequences have been extensively investigated by various authors [1], [2], [10], [11], [12] and [13], each approaching the concept from different angles. A spliced sequence essentially combines convergent sequences through the use of partitions. Yurdakadim et al. extended this idea by incorporating bounded sequences instead of strictly convergent ones, while Bartoszewicz et al. [1] further expanded the concept by incorporating the density of points.

In this paper, we delve into the realm of a double density of a given point in $\mathbb{R}$ and establish the relation between double density of a point and spliced double sequences and also examine limit points with density of a point.

Let us firstly give a number of elemantary notions and definitions that will be required along the paper.

Definition 1.1. A double sequence $[x] = (x_{jk})$ is said to be *Pringsheim convergent* to L (denoted by $P\text{-}\lim x = L$) if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ with $|x_{jk} - L| < \epsilon$ for all $j, k > N$. In this case, L is called the *Pringsheim limit* of $[x]$, and the space of all such sequences is denoted by $c^{(2)}$ [7].

Definition 1.2. A double sequence (x_{jk}) is said to be bounded in the sense of Pringsheim if there is a positive number M with $|x_{jk}| < M$ for all $j, k \in \mathbb{N}$. The norm of $[x]$ is defined as

$$\|x\|_{\infty,(2)} := \sup_{j,k} |x_{jk}|,$$

the space of all such sequences is denoted by $l_\infty^{(2)}$ [6], [8].

Definition 1.3. Let $A = \left(a_{jk}^{nm}\right)$ be a four-dimensional summability matrix and $[x] = (x_{jk})$ be a double sequence. If the transformed sequence $[Ax] := \{(Ax)_{nm}\}$ is Pringsheim convergent to L, then $[x]$ is said to be *A-summable* to L, where

$$(Ax)_{nm} := \sum_{j,k} a_{jk}^{nm} x_{jk}, \quad \text{for any } n, m \in \mathbb{N} \text{ [3]}.$$

A four-dimensional matrix A is called *RH-regular* if it transforms every bounded Pringsheim convergent sequence into a Pringsheim convergent sequence with the same Pringsheim limit [3].

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Let $E \subset \mathbb{N} \times \mathbb{N}$ and let A be an RH-regular matrix. If the limit

$$P\text{-}\lim_{n,m} \sum_{(j,k)\in E} a_{jk}^{nm}$$

exists, then the set E is said to have A-*double density*, denoted by $\delta_A^{(2)}(E)$.

The four-dimensional Cesàro matrix $(C,1,1) = \left(c_{jk}^{nm}\right)$ is defined by

$$c_{jk}^{nm} = \begin{cases} \frac{1}{nm}, & \text{if } j \le n \text{ and } k \le m, \\ 0, & \text{otherwise.} \end{cases}$$

In the definition of A-double density, if we take $A = (C,1,1)$, then the *natural double density* of E is obtained, denoted by $\delta^{(2)}(E)$ [4].

Definition 1.4. Let A be a non-negative RH-regular matrix and $E \subset \mathbb{N} \times \mathbb{N}$. The quantities

$$\overline{\delta_A^{(2)}}(E) := \limsup_{n,m} \sum_{(j,k)\in E} a_{jk}^{nm}, \qquad \underline{\delta_A^{(2)}}(E) := \liminf_{n,m} \sum_{(j,k)\in E} a_{jk}^{nm}$$

are called the A-*double upper density* and A-*double lower density* of E, respectively. If these two values are equal, then the common value is called the A-*double density* of E, denoted by $\delta_A^{(2)}(E)$.

If we take $A = (C,1,1)$ specially, we get $d^{(2)}$ instead of $\delta_A^{(2)}$.

Let $(x_{jk}) \in \ell_\infty^{(2)}$ and $y \in \mathbb{R}$. The A-double upper and lower density of the point y are defined by

$$\overline{\delta_A^{(2)}}(y) := \lim_{\epsilon\to 0^+} \overline{\delta_A^{(2)}}\left(\{(j,k) : |x_{jk} - y| \le \epsilon\}\right),$$

$$\underline{\delta_A^{(2)}}(y) := \lim_{\epsilon\to 0^+} \underline{\delta_A^{(2)}}\left(\{(j,k) : |x_{jk} - y| \le \epsilon\}\right).$$

If these two quantities are equal, then the common value is called the A-*double density of the point* y and is denoted by $\delta_A^{(2)}(y)$.

2. Main Results

In this section we give main results.

Theorem 2.1. *Let $\delta_A^{(2)}(y)$ exist for every $y \in \mathbb{R}$. Then the set $D = \{y \in \mathbb{R} : \delta_A^{(2)}(y) > 0\}$ is countable and $\sum_{y\in D} \delta_A^{(2)}(y) \le 1$.*

Proof. Let (r_s) be a strictly decreasing sequence such that $r_s \to 1$. For a fixed $v \in \mathbb{N}$, define the set

$$D_v = \left\{ y \in \mathbb{R} : \delta_A^{(2)}(y) \ge \frac{1}{v} \right\}.$$

Let $y_1, y_2, \ldots, y_l \in D_v$ be distinct elements. Set

$$\varepsilon = \min_{\mu \ne u} \frac{|y_\mu - y_u|}{3} > 0,$$

and define the sets

$$E_\mu = \{(j,k) : |x_{jk} - y_\mu| \le \varepsilon\}.$$

Then the sets $E_1, E_2, \ldots, E_l$ are pairwise disjoint and satisfy $\delta_A^{(2)}(E_\mu) \ge \frac{1}{v}$ for each $\mu = 1, \ldots, l$.

Since $A=(a_{jk}^{nm})$ is an RH-regular matrix, there exists $n_0\in\mathbb{N}$ with for any fixed $\tau>0$ and for all $n,m\geq n_0$,

$$\sum_{(j,k)\in E_\mu} a_{jk}^{nm}\geq \frac{1}{v}-\tau \quad \text{and} \quad \sum_{(j,k)=(1,1)}^{\infty,\infty} a_{jk}^{nm}\leq r_p$$

for each $\mu=1,\ldots,l$, where p is fixed. Since the sets $E_1,\ldots,E_l$ are disjoint, we have

$$\sum_{(j,k)\in\bigcup_{\mu=1}^{l}E_\mu} a_{jk}^{nm}=\sum_{\mu=1}^{l}\sum_{(j,k)\in E_\mu} a_{jk}^{nm}\geq \frac{l}{v}-l\tau.$$

Hence,

$$r_s\geq \frac{l(1-v\tau)}{v} \quad\Rightarrow\quad \frac{vr_s}{1-v\tau}\geq l.$$

This implies that D_v must be finite. Therefore,

$$\sum_{y\in D_v}\delta_A^{(2)}(y)\leq r_s.$$

Since $D_1\subset D_2\subset D_3\subset\ldots$ and $D=\bigcup_{v=1}^{\infty}D_v$, we have

$$\sum_{y\in D}\delta_A^{(2)}(y)=\lim_{v\to\infty}\sum_{y\in D_v}\delta_A^{(2)}(y)\leq r_s.$$

Taking the limit as $r_s\to 1$, we obtain

$$\sum_{y\in D}\delta_A^{(2)}(y)\leq 1.$$

Thus, the set D must be at most countable. $\square$

Theorem 2.2. *There is a bounded sequence (x_{jk}) such that $\overline{\delta_A^{(2)}}(\{(j,k):|x_{jk}-y|\leq\epsilon\})=\overline{d^{(2)}}(\{(j,k):|x_{jk}-y|\leq\epsilon\})=1$ for all $\epsilon>0$ and every $y\in[0,1]$ where A is the four dimensional Cesaro matrix.*

Proof. Let (z_{jk}) be a sequence such that its set of limit points is $[0,1]$. Such a sequence can be obtained by repeating infinite times with of all rational number in the range $[0,1]$ where not in the same row and same column. $\square$

Theorem 2.3. *Let (x_{jk}) be a bounded sequence, $\delta_A^{(2)}(y)$ exist for any $y\in\mathbb{R}$ and $\sum_{y\in D}\delta_A^{(2)}(y)=1$. Then*

$$\lim_{n,m\to\infty}(Ax)_{nm}=\sum_{y\in D}\delta_A^{(2)}(y)\,.y.$$

Proof. Since (x_{jk}) is bounded, there exists $M>0$ with $|x_{jk}|\leq M$ for all $(j,k)\in\mathbb{N}\times\mathbb{N}$. Let $D=\{y_\mu\}_{\mu\in\mathbb{N}}$ be the set of distinct points.

Given $\epsilon>0$, choose $r\in\mathbb{N}$ with

$$\sum_{\mu=1}^{r}\delta_A^{(2)}(y_\mu)>1-\epsilon \quad\text{and}\quad \left|\sum_{\mu=r+1}^{\infty}\delta_A^{(2)}(y_\mu)y_\mu\right|<\epsilon.$$

Let

$$\eta:=\frac{1}{N}<\frac{1}{3}\min_{1\leq\mu\neq u\leq r}|y_\mu-y_u|.$$

Define the sets

$$E_\mu=\{(j,k):|x_{jk}-y_\mu|<\eta\},\quad \mu=1,\ldots,r.$$

Then $E_1, \dots, E_r$ are pairwise disjoint. Moreover, we get

$$\delta_A^{(2)}(y_\mu) - \frac{\epsilon}{r(M+1)} \le \underline{\delta}_A^{(2)}(E_\mu) \le \overline{\delta}_A^{(2)}(E_\mu) \le \delta_A^{(2)}(y_\mu) + \frac{\epsilon}{r(M+1)}$$

for $\mu = 1, \dots, r$.

Since $A = (a_{jk}^{nm})$ is RH-regular, there is $N_0 \in \mathbb{N}$ with for each $n, m \ge N_0$ and $\mu = 1, \dots, r$,

$$\underline{\delta}_A^{(2)}(E_\mu) - \eta < \sum_{(j,k)\in E_\mu} a_{jk}^{nm} < \overline{\delta}_A^{(2)}(E_\mu) + \eta.$$

It follows that

$$\left| \sum_{(j,k)\in E_\mu} a_{jk}^{nm} - \delta_A^{(2)}(y_\mu) \right| < \eta + \frac{\epsilon}{r(M+1)}. \tag{1}$$

Now consider

$$(Ax)_{nm} = \sum_{j,k} a_{jk}^{nm} x_{jk}.$$

We estimate this expression from above:

$$\begin{aligned} (Ax)_{nm} \le \sum_{(j,k)\in E_1} & a_{jk}^{nm}(y_1 + \eta) + \dots + \sum_{(j,k)\in E_r} a_{jk}^{nm}(y_r + \eta) \\ & + \sum_{(j,k)\in (E_1\cup\dots\cup E_r)^c} a_{jk}^{nm} \cdot M. \end{aligned}$$

Since A is RH-regular, we can determine $N_1 \ge N_0$ with

$$\sum_{j,k} a_{jk}^{nm} < 1 + \epsilon \quad \text{for each } n, m \ge N_1.$$

Then,

$$\sum_{(j,k)\in (E_1\cup\dots\cup E_r)^c} a_{jk}^{nm} < (1+\epsilon) - \sum_{\mu=1}^{r} \sum_{(j,k)\in E_\mu} a_{jk}^{nm}.$$

On the other hand, from the earlier estimate:

$$\sum_{\mu=1}^{r} \sum_{(j,k)\in E_\mu} a_{jk}^{nm} > \sum_{\mu=1}^{r} \delta_A^{(2)}(y_\mu) - \frac{r}{N} - \frac{\epsilon}{M+1} > 1 - \frac{r}{N} - \left(1 + \frac{1}{M+1}\right)\epsilon.$$

Combining these gives:

$$\sum_{(j,k)\in (E_1\cup\dots\cup E_r)^c} a_{jk}^{nm} < \frac{r}{N} + \left(2 + \frac{1}{M+1}\right)\epsilon.$$

Hence, for all $n, m \ge N_1$:

$$(Ax)_{nm} \le \sum_{\mu=1}^{r} \sum_{(j,k)\in E_\mu} a_{jk}^{nm}(y_\mu + \eta) + \frac{Mr}{N} + \left(2 + \frac{1}{M+1}\right) M\epsilon,$$

$$(Ax)_{nm} \ge \sum_{\mu=1}^{r} \sum_{(j,k)\in E_\mu} a_{jk}^{nm}(y_\mu - \eta) - \frac{Mr}{N} - \left(2 + \frac{1}{M+1}\right) M\epsilon.$$

Therefore, we obtain:

$$\left| (Ax)_{nm} - \sum_{\mu=1}^{r} \sum_{(j,k)\in E_\mu} a_{jk}^{nm} y_\mu \right| \le \frac{Mr}{N} + \left(2 + \frac{1}{M+1}\right) M\epsilon. \tag{2}$$

Using (1) and (2), for $n, m \geq N_1$, we estimate:

$$\left|(Ax)_{nm} - \sum_{\mu=1}^{\infty} \delta_A^{(2)}(y_\mu) y_\mu\right| \leq \left|(Ax)_{nm} - \sum_{\mu=1}^{r} \delta_A^{(2)}(y_\mu) y_\mu\right| + \left|\sum_{\mu=r+1}^{\infty} \delta_A^{(2)}(y_\mu) y_\mu\right|$$

$$\leq \sum_{\mu=1}^{r} \left|\sum_{(j,k)\in E_\mu} a_{jk}^{nm} - \delta_A^{(2)}(y_\mu)\right| \cdot (|y_\mu| + \eta)$$

$$+ \eta + \frac{Mr}{N} + \left(2M + \frac{M}{M+1} + 1\right)\epsilon$$

$$\leq r\left(\frac{1}{N} + \frac{\epsilon}{r(M+1)}\right)(M + 1 + \frac{1}{N}) + \frac{1}{N} + \frac{Mr}{N} + \left(2M + \frac{M}{M+1} + 1\right)\epsilon.$$

Since N can be determined arbitrarily large for any fixed $\epsilon > 0$, we conclude:

$$\left|(Ax)_{nm} - \sum_{\mu=1}^{\infty} \delta_A^{(2)}(y_\mu) y_\mu\right| \leq \left(2M + \frac{M}{M+1} + 2\right)\epsilon.$$

This shows that

$$\lim_{n,m\to\infty} (Ax)_{nm} = \sum_{\mu=1}^{\infty} \delta_A^{(2)}(y_\mu) y_\mu.$$

□

The next definition has been given in [9]

Definition 2.4. Let $\{E_i\}_{i\in\mathbb{N}}$ be a fixed infinite partition of $\mathbb{N}\times\mathbb{N}$. For each $i \in \mathbb{N}$, let $\gamma^{(i)} = \left(\gamma_{j,k}^{(i)}\right)_{j,k\in\mathbb{N}}$ be a double sequence such that $P\text{-}\lim_{j,k} \gamma_{j,k}^{(i)} = \Gamma^{(i)}$.

Then, the splice of sequences $\gamma^{(i)}$ over the infinite partition $\{E_i\}$ is the double sequence $x = (x_{r,s})_{r,s\in\mathbb{N}}$ defined by:

$$(r,s) \in E_i \quad \Rightarrow \quad x_{r,s} := \gamma_{j,k}^{(i)} \quad \text{for some } j,k \in \mathbb{N} \text{ such that } (r,s) = (\nu_i(j), \mu_i(k)),$$

where $\nu_i : \mathbb{N} \to \mathbb{N}$ and $\mu_i : \mathbb{N} \to \mathbb{N}$ are suitable bijections associated with the index transformation on E_i.

The following theorem shows that double version of Osikiewicz theorem implies the assumptions of the above theorem.

Theorem 2.5. *Let (x_{jk}) be a spliced sequence over a partition $\{E_i\}$, $y_i = \sum_{\substack{(j,k)\in E_i \\ j,k\to\infty}} x_{jk}$, $\delta_A^{(2)}(E_i)$ exists for each i and $\sum_i \delta_A^{(2)}(E_i) = 1$. Then $\delta_A^{(2)}(y)$ exists for all $y \in \mathbb{R}$ and $\delta_A^{(2)}(y_i) = \delta_A^{(2)}(E_i)$.*

Proof. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ with

$$\sum_{i=1}^{N} \delta_A^{(2)}(E_i) > 1 - \epsilon.$$

Let $\alpha > 0$ be with the open intervals $(y_i - \alpha, y_i + \alpha)$ are pairwise disjoint for $i = 1, 2, \ldots, N$.

Consider the set

$$\{(j,k) : x_{jk} \in (y_i - \alpha, y_i + \alpha)\} \setminus E_i.$$

This set contains at most a finite number of rows and columns. Hence, we have

$$\delta_A^{(2)}(\{(j,k) : x_{jk} \in (y_i - \alpha, y_i + \alpha)\}) \supseteq \delta_A^{(2)}(E_i),$$

and consequently,

$$\delta_A^{(2)}(y_i) \geq \delta_A^{(2)}(E_i).$$

On the other hand, we estimate the upper density:

$$\begin{aligned}
\overline{\delta_A^{(2)}}(y_i) &= \lim_{\beta\to 0^+} \overline{\delta_A^{(2)}}\left(\{(j,k) : x_{jk} \in (y_i-\beta, y_i+\beta)\}\right) \\
&= 1 - \lim_{\beta\to 0^+} \underline{\delta_A^{(2)}}\left(\{(j,k) : x_{jk} \in (y_i-\beta, y_i+\beta)\}\right) \\
&\leq 1 - \underline{\delta_A^{(2)}}\left(\{(j,k) : x_{jk} \in (y_i-\beta, y_i+\beta)\}\right) \\
&\leq 1 - \sum_{\substack{v=1 \\ v\neq i}}^{N} \delta_A^{(2)}(E_v) \\
&< \delta_A^{(2)}(E_i) + \epsilon.
\end{aligned}$$

Therefore, combining both estimates, we conclude:

$$\delta_A^{(2)}(y_i) = \delta_A^{(2)}(E_i).$$

Now, suppose $y \notin \{y_i\}_{i=1}^{\infty}$. Then for given $\epsilon > 0$, we can choose N with

$$\sum_{i=1}^{N} \delta_A^{(2)}(E_i) > 1 - \epsilon.$$

Let

$$\alpha := \operatorname{dist}\left(y, \{y_1, \ldots, y_N\}\right),$$

and observe that the open ball $B(y, \alpha/2)$ does not intersect any of the intervals $(y_i - \alpha, y_i + \alpha)$ for $i = 1, \ldots, N$. Therefore,

$$\overline{\delta_A^{(2)}}\left(\left\{(j,k) : |x_{jk} - y| < \frac{\alpha}{2}\right\}\right) < \epsilon,$$

which implies

$$\delta_A^{(2)}(y) = 0.$$

□

Theorem 2.6. *Let (x_{jk}) be a bounded sequence. If $\overline{\delta_A^{(2)}}(y) = 1$, then y is a limit point of the sequence $(Ax)_{nm}$.*

Proof. Since (x_{jk}) is a bounded sequence, there exists a constant $M > 0$ with $|x_{jk}| \leq M$ for each $(j,k) \in \mathbb{N} \times \mathbb{N}$. Let $y \in \mathbb{R}$ and assume that $\overline{\delta_A^{(2)}}(y) = 1$.

Fix $N \in \mathbb{N}$ and define the set

$$E_N = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : |x_{jk} - y| < \frac{1}{N}\right\}.$$

By the definition of $\overline{\delta_A^{(2)}}(y)$, there exist integers $n_N, m_N \geq N$ such that

$$\sum_{(j,k)\in E_N} a_{jk}^{n_N m_N} > \overline{\delta_A^{(2)}}(E_N) - \frac{1}{N} = 1 - \frac{1}{N},$$

and by the RH-regularity of A, we also have

$$\sum_{(j,k)} a_{jk}^{n_N m_N} < 1 + \frac{1}{N}.$$

Note that for $(j,k) \in E_N$, we have

$$y - \frac{1}{N} < x_{jk} < y + \frac{1}{N},$$

and for $(j,k) \notin E_N$, it holds that

$$-M \leq x_{jk} \leq M.$$

Hence, we can estimate the transformed sum as

$$\begin{aligned}\sum_{(j,k)\in E_N} a_{jk}^{n_N m_N}(y - \tfrac{1}{N}) - \sum_{(j,k)\notin E_N} a_{jk}^{n_N m_N} M &\leq \sum_{(j,k)} a_{jk}^{n_N m_N} x_{jk} \\ \leq \sum_{(j,k)\in E_N} a_{jk}^{n_N m_N}(y + \tfrac{1}{N}) + \sum_{(j,k)\notin E_N} a_{jk}^{n_N m_N} M.\end{aligned}$$

Subtracting y from all parts and rearranging, we obtain

$$\begin{aligned}&y\left(\sum_{(j,k)} a_{jk}^{n_N m_N} - 1\right) - \sum_{(j,k)\in E_N} a_{jk}^{n_N m_N}\tfrac{1}{N} - \sum_{(j,k)\notin E_N} a_{jk}^{n_N m_N}(M+y) \\ &\leq \sum_{(j,k)} a_{jk}^{n_N m_N} x_{jk} - y \\ &\leq \sum_{(j,k)\notin E_N} a_{jk}^{n_N m_N}\tfrac{1}{N} + \sum_{(j,k)\notin E_N} a_{jk}^{n_N m_N}(M-y) + y\left(\sum_{(j,k)} a_{jk}^{n_N m_N} - 1\right).\end{aligned}$$

Taking absolute values yields

$$\left|\sum_{(j,k)} a_{jk}^{n_N m_N} x_{jk} - y\right| \leq \left(\sum_{(j,k)\notin E_N} a_{jk}^{n_N m_N}\right)\cdot\left(\frac{1}{N} + M + |y|\right) + \frac{|y|}{N}.$$

Moreover, since

$$\sum_{(j,k)\notin E_N} a_{jk}^{n_N m_N} = \sum_{(j,k)} a_{jk}^{n_N m_N} - \sum_{(j,k)\in E_N} a_{jk}^{n_N m_N} < \left(1 + \frac{1}{N}\right) - \left(1 - \frac{1}{N}\right) = \frac{2}{N},$$

we finally deduce

$$\begin{aligned}|(Ax)_{n_N m_N} - y| &= \left|\sum_{(j,k)} a_{jk}^{n_N m_N} x_{jk} - y\right| \\ &\leq \frac{2}{N}(M + |y|) + \frac{|y|+1}{N} + \frac{1}{N^2}.\end{aligned}$$

Since the right-hand side tends to zero as $N \to \infty$, it follows that y is a limit point of the double sequence $\{(Ax)_{nm}\}$. □

Theorem 2.7. *Let (x_{jk}) be a bounded sequence and let y, z $(y \neq z)$ be given such that $\overline{\delta_A^{(2)}}(y) = \overline{\delta_A^{(2)}}(z) = 1$. Then the limit of the double sequence $\{(Ax)_{nm}\}$ does not exist.*

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