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SOME IDENTITIES FOR TENTH ORDER MOCK THETA FUNCTIONS

SWAYAMPRABHA TIWARI, SAMEENA SABA, AND AYHAN ESI

Abstract. In this paper two lemmas for arbitrary general functions are given and then, by iteration,we express them as a continued fraction after that by specializing the parameters the continued fraction representation for the generalized tenth order mock theta functions are obtained.

1. INTRODUCTION

In his last letter to G .H. Hardy [9], S. Ramanujan listed seventeen mock theta functions of order three, five and seven. According to Ramanujan a mock theta function is a function $f(q), |q| < 1$, satisfying the following two conditions:

(0) For every root of unity ξ, there is a θ-function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially.

(1) There is no single θ-function which works for all ξ, i.e., for every θ-function $\theta(q)$ there is some root of unity ξ for which $f(q) - \theta(q)$ is unbounded as $q \to \xi$ radially.

The "lost notebook" contains eight identities involving four tenth order mock theta functions. In which Choi [1] has proved the first two of Ramanujan's tenth order mock theta functions, mock theta function identities and said that further identities will be proved in subsequent papers.

The Tenth Order Mock Theta Functions are:

$$\varphi(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q^2)_{n+1}}$$

$$\psi(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}}}{(q;q^2)_{n+1}}$$

$$X(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}$$

and

$$\chi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}$$

Generalization of Tenth Order Mock Theta Functions are:

$$\varphi(t,\alpha,z;q) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)nq^{\frac{n(n-5)}{2}+n\alpha}}{(z^2/q;q^{2n})_{n+1}}$$

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$$\psi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n-3)}{2}+n\alpha} z^{2n+1}}{(z^2/q;q^2)_{n+1}}$$

$$X(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n q^{n(n-2)+n\alpha} z^{n}}{(-z^2/q;q)_{2n}}$$

and

$$\chi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n q^{n^2+n\alpha} z^{n+1}}{(-z^2/q;q)_{2n+1}}$$

Taking $t=0, \alpha=1, z=q$ in above generalized tenth order mock theta functions, the original tenth order mock theta functions of Ramanujan are obtained [7].

2. NOTATIONS

The following usual basic hypergeometric notations will be used:

$$\left(a;q^k\right)_n = \prod_{m=1}^{n}\left(1-aq^{k(m-1)}\right), \left|q^k\right|<1, \text{ n, non-negative integer}$$

$$\left(a;q^k\right)_0 = 1$$

$$\left(a;q^k\right)_\infty = \prod_{m=1}^{n}\left(1-aq^{k(m-1)}\right)$$

$$\left(a_1,a_2,a_3 \ldots\ldots a_m;q^k\right)_n = \left(a_1;q^k\right)_n\left(a_2;q^k\right)_n \cdots\cdots \left(a_m;q^k\right)_n$$

A generalized basic hypergeometric function with base q is defined as:

$$\dot{r}\phi_s\left[\begin{matrix} a_1,a_2,\ldots\ldots a_r; \\ b_1,b_2,\ldots\ldots,b_s;q,z\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(a_1;q)_n \ldots\ldots (a_r;q)_n z^n}{(b_1;q)_n \ldots\ldots (b_s;q)_n (q;q)_n}\left[(-1)^n q^{\frac{n^2-n}{2}}\right]^{1+s-r} z^n$$

For $r=s+1$, the above series is convergent for $|z|<1$, for $r\le s$, the above series is convergent for all z and for $r>s+1$, the diverges for all z except $z=0$.

3. SOME LEMMAS FOR GENERALIZED TENTH ORDER MOCK THETA FUNCTION

In this section the two lemmas for general functions will be given and proved as follows: Let

$$\mathrm{A}(t,\alpha,z,\beta;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n q^{n^2-2n+n\alpha} z^n \beta^{2n}}{\left(\frac{-\beta z^2}{q};q\right)_{2n}} \tag{3.1}$$

$$\mathrm{B}(t,\alpha,z,\beta;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n^2-5n}{2}+n\alpha} z^{2n}\beta^n}{\left(\frac{z^2}{q};q^2\right)_{n+1}} \tag{3.2}$$

Taking $\beta=1$ in eq^n (3.1),

$$\mathrm{A}(t,\alpha,z,1;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n q^{n^2-2n+n\alpha} z^n}{\left(\frac{-z^2}{q};q\right)_{2n}}$$

$$\mathrm{A}(t,\alpha,z,1;q) = X(t,\alpha,z;q)$$

Taking $\beta=q$ in eq^n (3.1),

$$\begin{aligned}
\mathrm{A}(t,\alpha,z,q;q) &= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2+n\alpha}z^n}{(-z^2;q)_{2n}} \\
&= \frac{\left(1+\frac{z^2}{q}\right)}{z}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2+n\alpha}z^{n+1}}{\left(\frac{-z^2}{q};q\right)_{2n+1}}
\end{aligned}$$

and

$$z\ \mathrm{A}(t,\alpha,z,q;q) = \left(1+\frac{z^2}{q}\right)\chi(t,\alpha,z;q)$$

Taking $\beta = 1$ in eq^n (3.2),

$$\mathrm{B}(t,\alpha,z,1;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n^2-5n}{2}+n\alpha}z^{2n}}{\left(\frac{z^2}{q};q^2\right)_{n+1}}$$

$$\mathrm{B}(t,\alpha,z,1;q) = \varphi(t,\alpha,z;q)$$

Taking $\beta = q$ in eq^n (3.2),

$$\mathrm{B}(t,\alpha,z,q;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n^2-3n}{2}+n\alpha}z^{2n}}{\left(\frac{z^2}{q};q^2\right)_{n+1}}$$

$$z\ \mathrm{B}(t,\alpha,z,q;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n^2-3n}{2}+n\alpha}z^{2n+1}}{\left(\frac{z^2}{q};q^2\right)_{n+1}}$$

$$z\ \mathrm{B}(t,\alpha,z,q;q) = \psi(t,\alpha,z;q)$$

The proofs for the following Lemma's according to [7] are as follows:

3.1. LEMMA 1.

$$\mathrm{A}(0,1,z,\beta;q) + \frac{\beta^2 z}{(1+\beta z^2/q)}\mathrm{A}(0,1,z,\beta q;q) - \frac{\beta^3 z^3}{(1+\beta z^2)(1+\beta z^2/q)}\mathrm{A}\left(0,1,z,\beta q^2;q\right) = 1$$

Proof

$$\begin{aligned}
\mathrm{A}(0,1,z,\beta;q) &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2-n}z^n\beta^{2n}}{(-\beta z^2/q;q)_{2n}} \\
&= 1-\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2-(n+1)}z^{n+1}\beta^{2n+2}}{(-\beta z^2/q;q)_{2n+2}} \\
&= 1-\frac{\beta^2 z}{(1+\beta z^2/q)}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}z^n\beta^{2n}}{(-\beta z^2;q)_{2n+1}}
\end{aligned}$$

So

$$\begin{aligned}
&\frac{\left(1+\beta z^2/q\right)}{\beta^2 z}[\ \mathrm{A}(0,1,z,\beta;q)-1]+\mathrm{A}(0,1,z,\beta q;q) \\
&=\beta z^2\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+3n}z^n\beta^{2n}}{(-\beta z^2;q)_{2n+1}} \\
&= \frac{\beta z}{(1+\beta z^2)}\mathrm{A}\left(0,1,z,\beta q^2;q\right)
\end{aligned}$$

After Simplification lemma 1 is obtained .

For $\beta = 1$,

$$\mathrm{A}(0,1,z,1;q) + \frac{z}{(1+z^2/q)}\mathrm{A}(0,1,z,q;q) - \frac{z^3}{(1+z^2)(1+z^2/q)}\mathrm{A}\left(0,1,z,q^2;q\right) = 1$$

3.2. LEMMA 2.

$$\mathrm{B}(0,1,z,\beta;q) = \frac{z^3\beta}{q}\ \mathrm{B}(0,1,z,\beta q;q) + \frac{z^2}{q}\ \mathrm{B}\left(0,1,z,\beta q^2;q\right) + 1$$

Proof

$$\begin{aligned}
z\ \mathrm{B}(0,1,z,\beta q;q) &= \sum_{n=0}^{\infty} \frac{q^{\frac{n^2-n}{2}} z^{2n}\beta^n}{(z^2/q;q^2)_{n+1}} \\
&= \frac{q}{\beta z^2}\sum_{n=1}^{\infty} \frac{q^{\frac{n^2-3n}{2}} z^{2n}\beta^n}{(z^2/q;q^2)_{n}} \\
&= \frac{q}{\beta z^2}\sum_{n=1}^{\infty} \frac{q^{\frac{n^2-3n}{2}} z^{2n}\beta^n}{(z^2/q;q^2)_{n+1}}\left(1 - z^2 q^{2n-1}\right) \\
&= \frac{q}{\beta z^2}\sum_{n=0}^{\infty} \frac{q^{\frac{n^2-3n}{2}} z^{2n}\beta^n}{(z^2/q;q^2)_{n+1}} - \frac{1}{\beta}\sum_{n=0}^{\infty} \frac{q^{\frac{n^2+n}{2}} z^{2n}\beta^n}{(z^2/q;q^2)_{n+1}} - \frac{q}{\beta z^2} \\
z\ \mathrm{B}(0,1,z,\beta q;q) &= \frac{q}{\beta z^2}\ \mathrm{B}(0,1,z,\beta;q) - \frac{1}{\beta}\mathrm{B}\left(0,1,z,\beta q^2;q\right) - \frac{q}{\beta z^2} \\
\mathrm{B}(0,1,z,\beta;q) &= \frac{\beta z^3}{q}\ \mathrm{B}(0,1,z,\beta q;q) + \frac{z^2}{q}\ \mathrm{B}\left(0,1,z,\beta q^2;q\right) + 1
\end{aligned}$$

which proves the lemma 2 .

For $\beta = 1$,

$$\mathrm{B}(0,1,z,1;q) = \frac{z^3}{q}\ \mathrm{B}(0,1,z,q;q) + \frac{z^2}{q}\ \mathrm{B}\left(0,1,z,q^2;q\right) + 1$$

4. CONTINUED FRACTION FOR GENERALIZED TENTH ORDER MOCK THETA FUNCTIONS

4.1. By Lemma 1

$$\mathrm{A}(0,1,z,\beta;q) + \frac{\beta^2 z}{(1+\beta z^2/q)}\mathrm{A}(0,1,z,\beta q;q) - \frac{\beta^3 z^3}{(1+\beta z^2)(1+\beta z^2/q)}\mathrm{A}\left(0,1,z,\beta q^2;q\right) = 1$$

now the continued fraction representation is given by

$$\begin{aligned}\frac{\mathrm{A}(0,1,z,\beta;q)}{\mathrm{A}(0,1,z,\beta q;q)} &= \mathrm{S}(0,1,z,\beta;q)+\frac{\mathrm{T}(0,1,z,\beta;q)}{\left(\frac{\mathrm{A}(0,1,z,\beta q;q)}{\mathrm{A}(0,1,z,\beta q^2;q)}\right)}\\ &= \mathrm{S}(0,1,z,\beta;q)+\frac{\mathrm{T}(0,1,z,\beta;q)}{\mathrm{S}(0,1,z,\beta q;q)+\frac{\mathrm{T}(0,1,z,\beta q;q)}{\left(\frac{\mathrm{A}(0,1,z,\beta q^2;q)}{\mathrm{A}(0,1,z,\beta q^3;q)}\right)}}\\ &= S(0,1,z,\beta;q)+\frac{T(0,1,z,\beta,q)}{S(0,1,z\beta q;q)+\frac{T(0,1,z,\beta q;q)}{S(0,1,z,\beta q^2;q)+\frac{T(0,1,z,\beta q^2;q)}{\left(\frac{A(0,1,z,\beta q^3;q)}{A(0,1,z,\beta q^4;q)}\right)}}}\end{aligned}$$

where

$$\mathrm{S}(0,1,z,\beta;q)=\left[\frac{1}{\mathrm{A}(0,1,z,\beta q;q)}-\frac{\beta^2 z}{(1+\beta z^2/q)}\right]$$

$$\mathrm{T}(0,1,z,\beta;q)=\frac{\beta^3 z^3}{(1+\beta z^2)\,(1+\beta z^2/q)}$$

For $\beta=1$, we have

$$\frac{\mathrm{A}(0,1,z,1;q)}{\mathrm{A}(0,1,z,q;q)}=S(0,1,z,1;q)+\frac{T(0,1,z,1;q)}{S(0,1,z,q;q)+\frac{T(0,1,z,q;q)}{S(0,1,z,q^2;q)+\frac{T(0,1,z,q^2;q)}{\left(\frac{A(0,1,z,q^3;q)}{A(0,1,z,q^4;q)}\right)}}}$$

But

$$\mathrm{A}(t,\alpha,z,1;q)=\mathrm{X}(t,\alpha,z;q)$$

$$z\mathrm{A}(t,\alpha,z,q;q)=\left(1+\frac{z^2}{q}\right)\chi(t,\alpha,z;q)$$

so the continued fraction representation for the generalized tenth order mock theta functions $\mathrm{X}(0,1,z;q)$ and $\chi(0,1,z;q)$ are given as

$$\begin{aligned}&\frac{z}{\left(1+\frac{z^2}{q}\right)}\frac{\mathrm{X}(0,1,z;q)}{\chi(0,1,z;q)}\\ &=\mathrm{S}(0,1,z,1;q)+\frac{\mathrm{T}(0,1,z,1;q)}{\mathrm{S}(0,1,z,q;q)+\frac{\mathrm{T}(0,1,z,q;q)}{\mathrm{S}(0,1,z,q^2;q)+\frac{\mathrm{T}(0,1,z,q^2;q)}{\left(\begin{array}{c}\frac{\mathrm{A}(0,1,z,q^3;q)}{\mathrm{A}(0,1,z,q^4;q)}\\ \vdots\end{array}\right)}}}.\end{aligned}$$

4.2. By Lemma 2.

$$\begin{aligned}\frac{\mathrm{B}(0,1,z,\beta;q)}{\mathrm{B}(0,1,z,\beta q;q)} &= \left[\frac{z^3\beta}{q}+\frac{1}{\mathrm{B}(0,1,z,\beta q;q)}\right]+\frac{z^2/q}{\frac{\mathrm{B}(0,1,z,\beta q;q)}{\mathrm{B}(0,1,z,\beta q^2;q)}}\\ &= \mathrm{S}(0,1,z,\beta;q)+\frac{z^2/q}{\mathrm{S}(0,1,z,\beta q;q)+\frac{z^2/q}{\left(\frac{\mathrm{B}(0,1,z,\beta q^2;q)}{\mathrm{B}(0,1,z,\beta q^3;q)}\right)}}\\ &= \mathrm{S}(0,1,z,\beta;q)+\frac{z^2/q}{\mathrm{S}(0,1,z,\beta q;q)+\frac{z^2/q}{\mathrm{S}(0,1,z,\beta q^2;q)+\frac{z^2/q}{\left(\frac{\mathrm{B}(0,1,z,\beta q^3;q)}{\mathrm{B}(0,1,z,\beta q^4;q)}\right)}}}\end{aligned}$$

where

$$\mathrm{S}(0,1,z,\beta;q)=\frac{z^3\beta}{q}+\frac{1}{\mathrm{B}(0,1,z,\beta q;q)}$$

Putting $\beta=1$, we have

$$\frac{\mathrm{B}(0,1,z,1;q)}{\mathrm{B}(0,1,z,q;q)}$$
$$=\mathrm{S}(0,1,z,1;q)+\cfrac{z^2/q}{\mathrm{S}(0,1,z,q;q)+\cfrac{z^2/q}{\mathrm{S}(0,1,z,q^2;q)+\cfrac{z^2/q}{\left(\frac{\mathrm{B}(0,1,z,q^3;q)}{\mathrm{B}(0,1,z,q^4;q)}\right)}}}.$$

But

$$\mathrm{B}(t,\alpha,z,1;q)=\varphi(t,\alpha,z;q)$$

and

$$z\ \mathrm{B}(t,\alpha,z,q;q)=\psi(t,\alpha,z;q)$$

so the continued fraction representation for the generalized tenth order mock theta functions $\varphi(0,1,z;q)$ and $\psi(0,1,z;q)$ are given as

$$\frac{z\varphi(0,1,z;q)}{\psi(0,1,z;1)}=S(0,1,z,1;q)+\cfrac{z^2/1}{S(0,1,z,1;q)+\cfrac{z^2/q}{S(0,1,z,q;q)+\cfrac{z^2/q}{\left(\begin{array}{c}\frac{B(0,1,z,q^3;q)}{B(0,1,z,q^4;q)}\\ \vdots\end{array}\right)}}}$$

5. CONCLUSIONS

In this result we obtained a new technique to find the continued fraction representation for the generalized tenth order mock theta functions by using the two lemmas for the general functions. Both the continued fractions obtained by the result are very helpful in knowing more about.These mysterious mock theta functions.Mock theta functions play a crucial role in theoretical physics,particularly in the study of quantum field theory and string theory.Their importance lies in their ability to provide insights into fundamental mathematical principles and their applications across different disciplines. This work is original and not previously published/submitted elsewhere.

References

[1] Choi Y.S., *Tenth order mock theta functions in Ramanujan's 'Lost' Notebook*, Invent. Math. **136** (1999), 497–569.

[2] Gasper G., Rahman M., *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[3] Rainville E.D., *Special Function*, Chelsea Publishing Company, Bronx, New York, 1960.

[4] Ramanujan S., *Collected Papers*, Cambridge University Press, 1972, reprinted Chelsea, New York, 1962.

[5] Rawat P. K., *On Generalized Mock Theta Functions of Tenth Order*, Trends in Sciences **19** (2022), no. 3, 2070.

[6] Saba S., *Generating functions for generalized mock theta functions*, Pure and Applied Mathematics Journal **2** (2013), no. 2, 62–70.

[7] Srivastava B., *Ramanujan's Mock Theta Functions*, Math. J. Okayama Univ. **47** (2005), 163–174.

[8] Srivastava B., *Ramanujan's fifth order and tenth order mock theta functions – a generalization*, Proyecciones Journal of Mathematics **34** (2015), no. 3, 277–296.

Swayamprabha Tiwari: swayam6288@gmail.com
Integral University_1, Department of Mathematics and Statistics, 226026, Lucknow, India.

Sameena Saba: sameena@iul.ac.in

Integral University_2, Department of Mathematics and Statistics, 226026, Lucknow, India.

Ayhan Esi: aesi23@hotmail.com
Malatya Turgut Ozal University_3, Department of Basic Eng. Sci. (Math. Sect.) and Nat. Sci. Faculty, 44100, Malatya, Turkiye.

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