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ON FRACTAL FAITHFULNESS AND FINE FRACTAL PROPERTIES OF RANDOM VARIABLES WITH INDEPENDENT $\widetilde{Q}$-SYMBOLS

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This paper is dedicated to the memory of academician V.S.Korolyuk

Abstract. We prove sufficient conditions for the faithfulness of the family of cylinders generated by $\widetilde{Q}$ -expansions for the Hausdorff-Besicovitch dimension calculation. We also represent the conjecture about necessary and sufficient conditions for such a family of cylinders to be faithful. Based on these new results we study fine fractal properties of random variables with independent $\widetilde{Q}$-digits and prove exact formulae for the calculation of the Hausdorff dimension of the corresponding probability measure.

1. Introduction

Hausdorff measures and the Hausdorff dimension are important tools in the study of fractals and singularly continuous probability measures. The determination or even estimation of the Hausdorff dimension of a set or a measure is the crucial problem in fractal analysis, and a lot of research papers were devoted to these problems. Because of this reason many interesting methods for the simplification of the procedure of the determination of the Hausdorff dimension were invented and developed during last 20 years. One approach to the simplification of the determination of the Hausdorff dimension consists in some restrictions of admissible coverings. This idea came from Besicovitch's works and has been used by Rogers and Taylor to construct comparable net measures as approximations of the Hausdorff measures. In this paper we develop this approach via construction of net coverings which lead to a special family of net measures which are more general that comparable ones. We shall discuss the notion of faithfulness and non-faithfulness of the family of cylinders generated by different systems of numerations for the Hausdorff dimension calculation.

Let us shortly recall that a family Φ of subsets from the unit interval is said to be fine family of coverings if for any subset $E \subset [0;1]$ and for any $\varepsilon > 0$ there exists an at most countable ε-covering $\{E_k\}$ of E with $E_k \in \Phi$. *The α-dimensional Hausdorff measure* of a set $E \subset [0,1]$ with respect to a given fine family of coverings Φ is defined by

$$H^\alpha(E,\Phi) = \lim_{\epsilon\to 0} \inf_{|E_j|\le\epsilon} \sum_j |E_j|^\alpha = \lim_{\epsilon\to 0} H^\alpha_\epsilon(E,\Phi),$$

where the infimum is taken over all at most countable ϵ-coverings $\{E_j\}$ of E, $E_j \in \Phi$. The nonnegative number

$$\dim_H(E,\Phi) = \inf\{\alpha :\ H^\alpha(E,\Phi) = 0\}$$

is called the Hausdorff dimension of the set $E \subset [0,1]$ w.r.t. the family Φ. If Φ is the family of all subsets of $[0,1]$, or Φ coincides with the family of all closed (open) subintervals of [0,1], then $\dim_H(E,\Phi)$ is equal to the classical Hausdorff dimension $\dim_H(E)$ of the subset $E \subset [0,1]$.

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A fine covering family Φ is said to be a *faithful family of coverings (non-faithful family of coverings)* for the Hausdorff dimension calculation on $[0,1]$ if

$$\dim_H(E,\Phi) = \dim_H(E), \quad \forall E \subseteq [0,1]$$

$$(\text{resp. } \exists E \subseteq [0,1] : \dim_H(E,\Phi) \neq \dim_H(E)).$$

It is clear that any family Φ of comparable net-coverings (i. e., net-coverings which generate comparable net-measures) is faithful. Conditions for Vitali coverings to be faithful were studied by many authors (see, e.g., [5, 8, 10] and references therein). First steps in this direction have been done by A. Besicovitch ([7]), who proved the faithfulness for the family of cylinders of a binary expansion. His result was extended by P. Billingsley ([8]) to the family of s-adic cylinders, by M. Pratsiovytyi ([19]) to the family of Q-S-cylinders, by S. Albeverio and G. Torbin ([5]) to the family of Q^*-cylinders for those matrices Q^* whose elements $p_{0k}, p_{(s-1)k}$ are bounded away from zero.

In all of these papers authors used the essentially the same approach to prove the faithfulness of the corresponding family of coverings: it has been shown that there exists positive constants C and $n_0 \in N$ such that for any $\varepsilon > 0$ and for any interval $(a;b)$ with $b-a<\varepsilon$ there exists at most n_0 cylinders from fine covering families which cover the interval (a,b) and their lengths do not exceed the value $C\cdot(b-a)$. It is rather obvious that such families Φ of cylinders generates comparable Hausdorff measures and, therefore, they are faithful for the Hausdorff dimesnion calculation. In [2] authors correctly mentioned that it was rather paradoxical that initial examples of non-faithful families of coverings appeared firstly in the two-dimensional case (as a result of active studies of self-affine sets during the last decade of XX century (see, e. g., [6])). The family of cylinders of the classical continued fraction expansion can probably be considered as the first (and rather unexpected) example of non-faithful one-dimensional net-family of coverings ([16]). By using the approach which has been invented by Yuval Peres to prove the non-faithfulness of the family of continued fraction cylinders ([16]), in [2] the non-faithfulness for the family $\Phi(Q_\infty)$ of cylinders of the Q_∞-expansion with polynomially decreasing elements $\{q_i\}$ has been proven. This shows, in particular, that the family of cylinders of the classical Lüroth expansion is non-faithful. In the papers [1] and [18] problems of faithfulness for the family of cylinders of Cantor series expansions were solved, and it has been shown in [22] that these results are of extreme importance for the fractal analysis of the set of essentially non-normal numbers.

In 2012 M.Ibragim and G.Torbin [12] developed a new method to prove faithfulness of the family of cylinders of Q^*-expansion for those matrices Q^* whose elements $p_{0k}, p_{(s-1)k}$ do not tend to zero "rather quickly". In particular, they proved the following result.

Theorem *Let* $q_k^* := \max\{q_{0k}, q_{1k}, \dots, q_{s-1k}\}$. *If*

$$\begin{cases} \lim\limits_{k\to\infty} \frac{\ln q_{0,k}}{\ln(q_1^* q_2^* \dots q_k^*)} = 0, \\ \lim\limits_{k\to\infty} \frac{\ln q_{s-1,k}}{\ln(q_1^* q_2^* \dots q_k^*)} = 0, \end{cases} \tag{1.1}$$

then

$$\dim_H(E) = \dim_H(E,\Phi(Q^*)), \quad \forall E \subset [0,1].$$

This theorem, being a generalization of [5], extended the family of faithful coverings generated by cylinders of Q^*-expansion. In particular, one can easily apply this theorem to prove faithfulness of the family of cylinders generated by the following matrix

$$Q^* = \begin{pmatrix} \frac{1}{10} & \dots & \frac{1}{10k} \dots \\ \frac{1}{2} - \frac{1}{10} & \dots & \frac{1}{2} - \frac{1}{10k} & \dots \\ \frac{1}{2} - \frac{1}{10} & \dots & \frac{1}{2} - \frac{1}{10k} & \dots \\ \frac{1}{10} & \dots & \frac{1}{10k} \dots \end{pmatrix}$$

On the other hand, if $p_{0k}, p_{(s-1)k}$ tend to zero "rather quickly" (let, for example, s=4, $q_{0k} = q_{3k} = \frac{1}{10^k}, \quad , q_{1k} = q_{2k} = \frac{1}{2} - \frac{1}{10^k}$), then the above Theorem does not work.

In the next section we shall develop new approach to prove faithfulness of families of coverings and prove essentially new sufficient conditions for Q^*-cylinders to be faithful (we do not need any information about boundedness from zero of elements q_{0k} and $q_{(s-1)k}$ or any information about the speed of their convergence to zero).

2. On sufficient conditions of fractal faithfulness for the family of cylinders of $\widetilde{Q}$-expansions

The notion of $\widetilde{Q}$-expansion is well known now (see, e.g., [4]). The following theorem gives a rather general sufficient condition for the family of cylinders of $\widetilde{Q}$-expansion to be faithful for the Hausdorff-Besicovitch dimension calculation.

Theorem 1. *Let $Q_n = q_1 \cdot q_2 \cdot ... \cdot q_n$, where $q_k := \max\limits_i \{q_{ik}\}$, $0 \le i \le n_k - 1$.*

Let's assume that the following conditions hold:

$$\lim_{k \to \infty} n_k \cdot Q_k^\delta = 0, \quad \forall \delta > 0; \tag{2.2}$$

$$\sum_{k=1}^{\infty} Q_k^\delta < +\infty, \quad \forall \delta > 0. \tag{2.3}$$

Then the family $\Phi(\widetilde{Q})$ of cylinders generated by $\widetilde{Q}$-expansion of real numbers is faithful for the Hausdorff dimension calculation on the unit interval, i.e.,

$$\dim_H(E) = \dim_H(E, \Phi(\widetilde{Q})), \quad \forall E \subset [0; 1].$$

Proof. Let E be an arbitrary subset of $[0; 1]$, let α and ε be arbitrary positive numbers, and δ be an arbitrary number from the interval $(0; \alpha)$. Let $\{E_j\}$ be an arbitrary ε-covering of the set E, $E_j = [a_j, b_j]$, $|E_j| < \varepsilon$. For any E_j there exists the cylinder of minimal rank m_j,

$$\Delta \alpha_1 \alpha_2 \ldots \alpha_{m_j},$$

which is entirely contained in $[a_j; b_j]$. Since any cylinder of the k-th rank is a union of cylinders of the next rank, then there are at most $2(n_{m_j} - 1)$ cylinders of rank m_j, belonging to $[a_j; b_j]$.

Let M_j be the family of all cylinders of rank m_j belonging to $[a_j; b_j]$. Then the α–volume of these cylinders does not exceed the value

$$\begin{aligned} 2(n_{m_j} - 1)|E_j|^\alpha &\le 2(n_{m_j} - 1) \cdot (q_1 \cdot q_2 \cdot ... \cdot q_{m_j})^\delta \cdot |E_j|^{\alpha - \delta} \\ &\le 2 n_{m_j} Q_{m_j}^\delta \cdot |E_j|^{\alpha - \delta} \le 2 C_0(\delta) \cdot |E_j|^{\alpha - \delta}. \end{aligned}$$

Taking into account the assumption 2.2 of the theorem, we deduce that there exists $C_0 = C_0(\delta)$ such that $n_{m_j} Q_{m_j}^\delta \le C_0(\delta)$, $\forall j \in N$.

Let $c_j := \min\{x : x \in M_j\}$, and let $d_j := \max\{x : x \in M_j\}$.

How to cover $[a_j; c_j]$ with cylinders? Let

$$\Delta_{\gamma_1 \gamma_2 \ldots \gamma_{m_j} \ldots \gamma_{m_j + s_j^{(1)}}}$$

be the cylinder of the minimum rank belonging to $[a_j; c_j]$. It is clear that there exist at most $n_{m_j + s_j^{(1)}}$ such cylinders. Let us denote by L_j^1 the union of such cylinders. Their α-volume does not exceed the value

$$\left(n_{m_j + s_j^{(1)}} \right) \cdot \max |\Delta_{\gamma_1 \gamma_2 \ldots \gamma_{m_j} \ldots \gamma_{m_j + s_j^{(1)} - 1}} \, i|^\alpha =$$

(where the maximum is taken for all such cylinders belonging to L_j^1)

$$= \left(n_{m_j+s_j^{(1)}}\right) \cdot \max\left(|\Delta_{\gamma_1\gamma_2\ldots\gamma_{m_j+s_j^{(1)}-1}\,i}|^{\delta}|\Delta_{\gamma_1\gamma_2\ldots\gamma_{m_j+s_j^{(1)}-1}\,i}|^{\alpha-\delta}\right) \le$$

$$\le n_{m_j+s_j^{(1)}} \cdot Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}} \cdot Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}} \cdot |E_j|^{\alpha-\delta} \le C_0(\delta) \cdot Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}} \cdot |E_j|^{\alpha-\delta}.$$

Taking into account the assumption 2.2 of the theorem, we deduce that there exists $C_1 = C_1(\delta)$ such that $n_{m_j+s_j^{(1)}} Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}} \le C_1(\delta),\ \forall j \in N$.

Let $c_j^{(1)} := \min\{x : x \in L_j^1\}$.

How to cover $[a_j; c_j^{(1)}]$ with cylinders? Let

$$\Delta_{\gamma_1\gamma_2\ldots\gamma_{m_j}\ldots\gamma_{m_j+s_j^{(1)}}\ldots\gamma_{m_j+s_j^{(1)}+s_j^{(2)}}}$$

be the cylinder of minimum rank contained in the $[a_j, c_j^{(1)}]$. It is clear that there exist at most $n_{m_j+s_j^{(1)}+s_j^{(2)}}$ such cylinders. Let us denote by L_j^2 the union of such cylinders. Their α-volume does not exceed

$$\left(n_{m_j+s_j^{(1)}+s_j^{(2)}}\right) \cdot \max|\Delta_{\gamma_1\gamma_2\ldots\gamma_{m_j+s_j^{(1)}+s_j^{(2)}-1}\,i}|^{\alpha} =$$

(where the maximum is taken for all such cylinders belonging to L_j^2)

$$= \left(n_{m_j+s_j^{(1)}+s_j^{(2)}}\right) \cdot \max\left(|\Delta_{\gamma_1\gamma_2\ldots\gamma_{m_j+s_j^{(1)}+s_j^{(2)}-1}}|^{\delta}|\Delta_{\gamma_1\gamma_2\ldots\gamma_{m_j+s_j^{(1)}+s_j^{(2)}-1}\,i}|^{\alpha-\delta}\right) \le$$

$$\le n_{m_j+s_j^{(1)}+s_j^{(2)}} \cdot Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}+s_j^{(2)}} \cdot Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}+s_j^{(2)}} \cdot |E_j|^{\alpha-\delta} \le C_1(\delta) \cdot Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}+s_j^{(2)}} \cdot |E_j|^{\alpha-\delta}.$$

Continuing this process by induction, we can get two possible situations:

a) $a_j \equiv c_j^{(k)}$ for some $k \in N$;

b) $a_j < c_j^{(k)}, \quad \forall k \in N$.

In case a) we get a finite number of cylinders that cover $[a_j, c_j]$. Their total α-volume does not exceed the value

$$C_1(\delta) \cdot |E_j|^{\alpha-\delta} \cdot \left(\sum_{k=1}^{\infty} Q_k^{\frac{\delta}{2}}\right).$$

In case b) we get a countable number of cylinders covering $[a_j, c_j]$. Their total α-volume does not exceed the value

$$C_1(\delta) \cdot |E_j|^{\alpha-\delta} \cdot \left(Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}} + Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}+s_j^{(2)}} + \ldots + Q^{\frac{\delta}{2}}_{m_j+s_j^{(1)}+s_j^{(2)}+\ldots+s_j^{(k)}} + \ldots\right)$$

$$\le C_1(\delta) \cdot |E_j|^{\alpha-\delta} \cdot \left(\sum_{k=1}^{\infty} Q_k^{\frac{\delta}{2}}\right).$$

In both cases, there is a finite or countable number of cylinders that covers $[a_j, c_j]$ and their α-volume does not exceed the value

$$C_1(\delta) \cdot |E_j|^{\alpha-\delta} \cdot \left(\sum_{k=1}^{\infty} Q_k^{\frac{\delta}{2}}\right).$$

The situation is similar with $[d_j, b_j]$: there is a finite or countable number of cylinders that cover $[d_j, b_j]$ and whose α-volume does not exceed the value

$$C_1(\delta) \cdot |E_j|^{\alpha-\delta} \cdot \left(\sum_{k=1}^{\infty} Q_k^{\frac{\delta}{2}} \right).$$

Therefore, the segment $[a_j, b_j]$ can be covered by a finite or countable number of $\widetilde{Q}$-cylinders, and the α-volume of this covering does not exceed the value

$$2C_0(\delta) \cdot |E_j|^{\alpha-\delta} + 2C_1(\delta) \cdot |E_j|^{\alpha-\delta} \cdot \left(\sum_{k=1}^{\infty} Q_k^{\frac{\delta}{2}} \right).$$

Let $C_2(\delta) := \max\{C_0(\delta); C_1(\delta)\}$. Then

$$2C_0(\delta) \cdot |E_j|^{\alpha-\delta} + 2C_1(\delta) \cdot |E_j|^{\alpha-\delta} \cdot \left(\sum_{k=1}^{\infty} Q_k^{\frac{\delta}{2}} \right) \leq$$

$$\leq 2C_2(\delta) \cdot |E_j|^{\alpha-\delta} \cdot \left(1 + \sum_{k=1}^{\infty} Q_k^{\frac{\delta}{2}} \right) = 2C_2(\delta) \cdot |E_j|^{\alpha-\delta} \cdot (1 + S(\delta))$$

where, $S(\delta) := \sum\limits_{k=1}^{\infty} Q_k^{\frac{\delta}{2}}$.

Therefore, for a given set E, for any $\varepsilon > 0$, for any $\alpha > 0$, for any $\delta \in (0, \alpha)$ and for any ε-covering of E by segments $\{E_j\}$, there exists the ε-covering of the set E by $\widetilde{Q}$-cylinders whose α-volume does not exceed the value

$$2C_2(\delta) \cdot |E_j|^{\alpha-\delta} \cdot (1 + S(\delta)).$$

Therefore, for an arbitrary ε-covering of the set E by segments $\{E_j\}$ we get

$$H_\varepsilon^\alpha(E, \Phi(\widetilde{Q})) \leq 2C_2(\delta) \cdot (1 + S(\delta)) \cdot \sum_j |E_j|^{\alpha-\delta}, \quad \forall \delta \in (0; \alpha).$$

Therefore,

$$H_\varepsilon^\alpha(E, \Phi(\widetilde{Q})) \leq 2C_2(\delta) \cdot (1 + S(\delta)) \cdot H_\varepsilon^{\alpha-\delta}(E), \quad \forall \varepsilon > 0,\ \forall \alpha > 0,\ \forall \delta \in (0; \alpha).$$

So,

$$H^\alpha(E, \Phi(\widetilde{Q})) \leq 2C_2 \cdot (1 + S(\delta)) \cdot H^{\alpha-\delta}(E), \quad \alpha > 0,\ \forall \delta \in (0; \alpha).$$

Therefore,

$$\dim_H(E, \Phi(\widetilde{Q})) \leq \dim_H(E) + \delta, \quad \forall \delta > 0.$$

Since this inequality holds for any arbitrary small δ, we obtain

$$\dim_H(E, \Phi(\widetilde{Q})) \leq \dim_H(E),\ \forall E.$$

It is clear that for any fine covering family Φ and for any subset E the following inequality holds

$$\dim_H(E) \leq \dim_H(E, \Phi).$$

So, for an arbitrary set $E \subset [0, 1]$ we get the desired equality

$$\dim_H(E) = \dim_H(E, \Phi(\widetilde{Q})).$$

Remark 1. If the sequence $\{n_k\}$ is bounded, then condition 2.3 implies condition 2.2. That is, if $\sum\limits_{k=1}^{\infty} Q_k^{\delta} < +\infty,\ \forall \delta > 0$, then the family $\Phi(\widetilde{Q})$ is faithful.

Remark 2. If $\{n_k\}$ is unbounded, then condition condition 2.3 does not imply condition 2.2.

Let us consider the relevant counterexample. Let $n_k = 2^{2^k}$, $q_{ik} = \frac{1}{2^{2^k}}$ for each $i \in \{0, 1, \ldots, n_k - 1\}$. Then, $Q_k = \frac{1}{2^{2^1}} \cdot \frac{1}{2^{2^2}} \cdot \ldots \cdot \frac{1}{2^{2^k}} = \frac{1}{2^{2^{k+1}-2}}$. It is obvious that, $\forall \delta > 0$ series $\sum\limits_{k=1}^{\infty} Q_k^\delta$ converges. On the other hand

$$n_k Q_k^\delta = 2^{2^k} \cdot \frac{1}{2^{(2^{k+1}-2)\delta}} = 2^{2^k - \delta(2^{k+1}-2)} \to +\infty, \ \forall \delta < \frac{1}{2}.$$

Remark 3. Condition condition 2.2 does not imply condition 2.3.

In order to show this fact, consider the following example. Let $n_k = 3$, $\forall k \in N$, and let us consider the following matrix $\widetilde{Q} = ||q_{ik}||$, where $i \in \{0, 1, 2\}$, such that:

$$q_{0k} = \begin{cases} \frac{1}{3}, & \text{if } k = 10^s, \ s \in N; \\ 1 - \frac{1}{2^k}, & \text{if } k \neq 10^s, \ s \in N; \end{cases}$$

$$q_{1k} = q_{2k} = \frac{1 - q_{0k}}{2} = \begin{cases} \frac{1}{3}, & \text{if } k = 10^s, \ s \in N; \\ \frac{1}{2^{k+1}}, & \text{if } k \neq 10^s, \ s \in N. \end{cases}$$

Let us estimate

$$Q_{10^k} = (1 - \frac{1}{2})\ (1 - \frac{1}{4}) \cdot \ldots \cdot (1 - \frac{1}{2^9}) \cdot \frac{1}{3} \cdot (1 - \frac{1}{2^{11}}) \cdot \ldots \cdot (1 - \frac{1}{2^{10^s - 1}}) <$$

$$< 1 \cdot 1 \cdot \ldots \cdot 1 \cdot \frac{1}{3} \cdot \ldots \cdot 1 \cdot \frac{1}{3} \cdot \ldots < \left(\frac{1}{3}\right)^k.$$

For any $k \in N$, there exists a unique $s = s(k)$ such that $10^{s-1} \leq k < 10^s$. It is clear that $s \to \infty$ as $k \to \infty$.

Therefore,

$$\lim_{k \to \infty} n_k Q_k^\delta = 0, \ \forall \delta > 0.$$

In order to prove the divergence of the series $\sum\limits_{k=1}^{\infty} Q_k^\delta$ for some δ we estimate Q_k. It is well known that:

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right) = a_0 > 0,$$

then,

$$\prod_{n \in F} \left(1 - \frac{1}{2^n}\right) \geq a_0, \ \forall F \subset N.$$

Therefore,

$$a_0 \leq \prod_{n \in F} \left(1 - \frac{1}{2^n}\right) \leq 1, \ \forall F \subset N.$$

So, $\forall k \in N$

$$a_0 \cdot \left(\frac{1}{3}\right)^{s-1} \leq Q_k = \prod_{j=1}^{k} q_{0j} < \left(\frac{1}{3}\right)^s,$$

where $s = s(k)$.

Then,

$$\sum_{k=1}^{\infty} Q_k^\delta \geq 9 \cdot a_0^\delta + 90 \cdot \left(\frac{a_0}{3}\right)^\delta + 900 \cdot \left(\frac{a_0}{9}\right)^\delta + \ldots + (10^k - 10^{k-1}) \cdot \left(\frac{a_0}{3}\right)^\delta + \ldots =$$

$$= 9 \cdot a_0^\delta \left(1 + \frac{10}{3^\delta} + \frac{10^2}{3^{2\delta}} + \frac{10^3}{3^{3\delta}} + \ldots + \frac{10^k}{3^{k\delta}} + \ldots\right) = +\infty, \forall\ 0 < \delta < 1$$

Therefore, $\sum\limits_{k=1}^{\infty} Q_k^\delta = +\infty,\ \forall \delta \in (0;1)$.

Conjecture. We believe that conditions 2.2 and 2.3 are also necessary conditions for the faithfulness of the family $\Phi(\widetilde{Q})$ for the Hausdorff-Besicovitch dimension calculation on the unit interval.

3. Fine fractal properties of probability measures with independent $\widetilde{Q}$-symbols

Let $\{\xi_k\}$ be a sequence of independent random variables taking values $0, 1, ..., n_k - 1$ with probabilities $p_{0k}, p_{1k}, \ldots, p_{n_k-1k}$ respectively ($p_{ik} \geq 0$, $\sum\limits_{i=0}^{n_k-1} p_{ik} = 1$). The random variable

$$\xi = \Delta^{\widetilde{Q}}_{\xi_1\xi_2\ldots\xi_k\ldots}$$

is said to be the random variable with independent $\widetilde{Q}$-symbols, and the corresponding probability measure ν_ξ is called a probability measure with independents $\widetilde{Q}$-symbols. The distribution ξ is completely determined by two matrices: $\widetilde{Q}$ and $\widetilde{P} = ||p_{ik}||$.

The Lebesgue structure of ν_ξ is well studied (see, e.g., [4]). It is known, in particular, that the distribution of ξ is of pure type.

Let υ be a continuous probability measure $[0, 1]$, and let Φ be the family of cylinders generated by some expansion. Then

$$H^\alpha(E, \upsilon, \Phi) = \lim_{\varepsilon\to 0} \left[\inf_{\upsilon(E_j)\leq\varepsilon} \left\{ \sum_j \upsilon^\alpha(E_j) \right\} \right] = \lim_{\varepsilon\to 0} H^\alpha_\varepsilon(E, \upsilon, \Phi),$$

where $E_j \in \Phi$, $\bigcup_j E_j \supset E$.

The number

$$\dim_\upsilon(E, \Phi) = \inf\{\alpha :\ H^\alpha(E, \upsilon, \Phi)\ = 0\}$$

is called the Hausdorff–Billingsley dimension of a set E w.r.t. υ and Φ.

Let us recall ([17]) that for a given probability measure μ the number

$$\dim_H \mu = \inf_{\mu_(E)=1} \{\dim_H E\}$$

is said to be the Hausdorff dimension of the measure μ.

For the formulation of the following theorem, we introduce the notation:

$$h_j := -\sum_{i=0}^{n_j-1} p_{ij} \ln p_{ij}, ; \quad H_n := \sum_{j=1}^{n} h_j.$$

$$b_j := -\sum_{i=0}^{n_j-1} p_{ij} \ln q_{ij}; \quad B_n := \sum_{j=1}^{n} b_j.$$

$$d_j := -b_j^2 + \sum_{i=0}^{n_j-1} p_{ij} \ln^2 q_{ij}.$$

$$l_j := -h_j^2 + \sum_{i=0}^{n_j-1} p_{ij} \ln^2 p_{ij}.$$

Theorem 2. *Let $q_k = \max_i q_{ik}$ and $Q_k = q_1 \cdot q_2 \cdot ... \cdot q_k$. Let us assume that conditions 2.2 and 2.3 hold and*

$$\sum_{j=1}^{\infty} \frac{d_j}{j^2} < +\infty; \tag{3.4}$$

$$\sum_{j=1}^{\infty} \frac{l_j}{j^2} < +\infty; \tag{3.5}$$

$$\varliminf_{n\to\infty} \frac{B_n}{n} > 0. \tag{3.6}$$

Then the Hausdorff dimension of the distribution of a random variable with independent $\widetilde{Q}$-symbols is equal to

$$\dim_H \nu_\xi = \varliminf_{n\to\infty} \frac{H_n}{B_n}.$$

Proof. Let $\Delta_n(x) = \Delta^{\widetilde{Q}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)}$ be the cylinder of rank n containing x. Let $\nu = \nu_\xi$ be the corresponding probability measure and let μ be the Lebesgue measure on $[0,1]$.

Then

$$\nu(\Delta_n(x)) = p_{\alpha_1(x)1} \cdot p_{\alpha_2(x)2} \cdot \ldots \cdot p_{\alpha_n(x)n},$$
$$\mu(\Delta_n(x)) = q_{\alpha_1(x)1} \cdot q_{\alpha_2(x)2} \cdot \ldots \cdot q_{\alpha_n(x)n}.$$

Let us consider

$$\frac{\ln \nu(\Delta_n(x))}{\ln \mu(\Delta_n(x))} = \frac{\sum\limits_{j=1}^{n} \ln p_{\alpha_j(x)j}}{\sum\limits_{j=1}^{n} \ln q_{\alpha_j(x)j}}.$$

If a real number $x = \Delta^{\widetilde{Q}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}$ is chosen randomly so that $P(\alpha_j(x) = i) = p_{ij}$ i.e., the distribution of the random variable x coincides with the initial probability measure ν, then $\{\eta_j\} = \{\eta_j(x)\} = \{\ln p_{\alpha_j(x)j}\}$ and $\{\psi_j\} = \{\psi_j(x)\} = \{\ln q_{\alpha_j(x)j}\}$ are sequences of independent random variables with the following distributions.

η_j	$\ln p_{0j}$	$\ln p_{1j}$	$\ldots$	$\ln p_{(n_k-1)j}$
	p_{0j}	p_{1j}	$\ldots$	$p_{(n_k-1)j}$

ψ_j	$\ln q_{0j}$	$\ln q_{1j}$	$\ldots$	$\ln q_{(n_k-1)j}$
	p_{0j}	p_{1j}	$\ldots$	$p_{(n_k-1)j}$

Consider the following values:

$$M\eta_j = -h_j = \sum_{i=0}^{n_j-1} p_{ij} \ln p_{ij}, \ |h_j| \le \ln(n_j),$$

$$M\eta_j^2 = \sum_{i=0}^{n_j-1} p_{ij} \ln^2 p_{ij}.$$

$$D(\eta_j) = M(\eta_j^2) - M^2(\eta_j) = l_j$$

From assumption 3.5 and the strong law of large numbers it follows that for ν-almost all $x \in [0,1]$ the following equality holds

$$\lim_{n\to\infty} \frac{(\eta_1 + \eta_2 + ... + \eta_n) - M(\eta_1 + \eta_2 + ... + \eta_n)}{n} = 0.$$

Let us remark that $M(\eta_1 + \eta_2 + ... + \eta_n) = -H_n$.

Let's apply strong law of large numbers to the sequence $\{\psi_j\}$. Consider the following values:

$$M\psi_j = -b_j = \sum_{i=0}^{s-1} p_{ij} \ln q_{ij}, \quad M\psi_j^2 = \sum_{i=0}^{s-1} p_{ij} \ln^2 q_{ij}.$$

It is clear that

$$D(\psi_j) = M(\psi_j^2) - M^2(\psi_j) = d_j$$

From assumption 3.4 and the strong law of large numbers it follows that for ν-almost all $x \in [0,1]$ the following equality holds

$$\lim_{n\to\infty} \frac{(\psi_1 + \psi_2 + ... + \psi_n) - M(\psi_1 + \psi_2 + ... + \psi_n)}{n} = 0.$$

Let us remark that $M(\psi_1 + \psi_2 + ... + \psi_n) = -B_n$.

Consider the set

$$A = \left\{ x : \lim_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\psi_1(x) + \psi_2(x) + ... + \psi_n(x)} - \frac{H_n}{B_n} \right) = 0 \right\} =$$

$$= \left\{ x : \lim_{n\to\infty} \frac{\left(\frac{\eta_1(x)+\eta_2(x)+...+\eta_n(x)+H_n}{n}\right) - \frac{H_n}{B_n}\left(\frac{\psi_1(x)+\psi_2(x)+...+\psi_n(x)+B_n}{n}\right)}{\left(\frac{\psi_1(x)+\psi_2(x)+...+\psi_n(x)+B_n}{n}\right) - \frac{B_n}{n}} = 0 \right\}.$$

The classical Gibbs inequality implies that $-\sum_{i=0}^{n_j-1} p_{ij} \ln p_{ij} \le -\sum_{i=0}^{n_j-1} p_{ij} \ln p_{ij}$ (proofs of the Gibbs inequality can be found in [9, 15]). That is $h_j \le b_j$, and hence $0 \le \frac{H_n}{B_n} = \frac{\sum_{j=1}^n h_j}{\sum_{j=1}^n b_j} \le 1$.

Since (by the assumption 3.6 of the theorem) $\underline{\lim}_{n\to\infty} \frac{B_n}{n} > 0$, then exists a constant $c_1 > 0$, such that inequality $\left|\frac{B_n}{n}\right| \ge c_1$ holds for all $n \in N$.

Therefore, for ν-almost all $x \in [0,1]$

$$\lim_{n\to\infty} \frac{\left(\frac{\eta_1(x)+\eta_2(x)+...+\eta_n(x)+H_n}{n}\right) - \frac{H_n}{B_n}\left(\frac{\psi_1(x)+\psi_2(x)+...+\psi_n(x)+B_n}{n}\right)}{\left(\frac{\psi_1(x)+\psi_2(x)+...+\psi_n(x)+B_n}{n}\right) - \frac{B_n}{n}} = 0.$$

So, $\nu(A) = 1$ and $\dim_\nu(A, \Phi) = 1$.

Let us consider the sets

$$A_1 = \left\{ x : \underline{\lim}_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\psi_1(x) + \psi_2(x) + ... + \psi_n(x)} - \frac{H_n}{B_n} \right) = 0 \right\};$$

$$A_2 = \left\{ x : \underline{\lim}_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\psi_1(x) + \psi_2(x) + ... + \psi_n(x)} \right) \le \underline{\lim}_{n\to\infty} \frac{H_n}{B_n} \right\} =$$

$$= \left\{ x : \underline{\lim}_{n\to\infty} \frac{\ln \nu(\Delta_n(x))}{\ln \mu(\Delta_n(x))} \le \underline{\lim}_{n\to\infty} \frac{H_n}{B_n} \right\};$$

$$A_3 = \left\{ x : \underline{\lim}_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\psi_1(x) + \psi_2(x) + ... + \psi_n(x)} \right) \ge \underline{\lim}_{n\to\infty} \frac{H_n}{B_n} \right\} =$$

$$= \left\{ x : \underline{\lim}_{n\to\infty} \frac{\ln \nu(\Delta_n(x))}{\ln \mu(\Delta_n(x))} \ge \underline{\lim}_{n\to\infty} \frac{H_n}{B_n} \right\}.$$

It is obvious that $A \subset A_1$. Let us prove the inclusion of sets $A_1 \subset A_3$ and $A \subset A_2$. It is known that

$$\underline{\lim}_{n\to\infty} (x_n - y_n) \le \underline{\lim}_{n\to\infty} (x_n) - \underline{\lim}_{n\to\infty} (y_n).$$

Then

$$\underline{\lim}_{n\to\infty}\left(\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}-\frac{H_n}{B_n}\right)\leq \quad \underline{\lim}_{n\to\infty}\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}-\underline{\lim}_{n\to\infty}\frac{H_n}{B_n}$$

If $x\in A_1$, then

$$\underline{\lim}_{n\to\infty}\left(\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}-\frac{H_n}{B_n}\right)=0.$$

Therefore,

$$0\leq \underline{\lim}_{n\to\infty}\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}-\underline{\lim}_{n\to\infty}\frac{H_n}{B_n}.$$

Then

$$\underline{\lim}_{n\to\infty}\frac{H_n}{B_n}\leq \underline{\lim}_{n\to\infty}\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}.$$

Thus, $x\in A_3$, and $A_1\subset A_3$.

If $x\in A$, then

$$\lim_{n\to\infty}\left(\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}-\frac{H_n}{B_n}\right)=0.$$

Then

$$\lim_{n\to\infty}\left(\frac{H_n}{B_n}-\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}\right)=0.$$

Therefore,

$$0=\underline{\lim}_{n\to\infty}\left(\frac{H_n}{B_n}-\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}\right)\leq \underline{\lim}_{n\to\infty}\frac{H_n}{B_n}-\underline{\lim}_{n\to\infty}\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}.$$

Then

$$\underline{\lim}_{n\to\infty}\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}\leq \underline{\lim}_{n\to\infty}\frac{H_n}{B_n}.$$

Thus, $x\in A_2$, and $A\subset A_2$.

From $A\subset A_2$, it follows that $\dim_\mu(A,\Phi)\leq\dim_\mu(A_2,\Phi)$. Let $D=\underline{\lim}_{n\to\infty}\frac{H_n}{B_n}$, from theorem [8], it follows that $\dim_\mu(A_2,\Phi)\leq D$, therefore, $\dim_\mu(A,\Phi)\leq D$.

From the condition $A_1\subset A_3$ and from the Billingsley theorem [8] we get $\dim_\mu(A,\Phi)\geq D\cdot\dim_\nu(A,\Phi)=D\cdot 1=D$. So, $\dim_\mu(A,\Phi)=D$.

Since μ is the Lebesgue measure on $[0,1]$, we get $\dim_H(A,\Phi)=\dim_\mu(A,\Phi)=D$.

From the above theorem, and from assumptions 2.2 and 2.3 of this theorem it follows that the family $\Phi(\widetilde{Q})$ of cylinders of the $\widetilde{Q}$-expansion is faithful for the Hausdorff–Besicovitch dimension calculation on the unit interval $[0,1]$. Hence,

$$\dim_H(A,\Phi)=\dim_H(A)=D.$$

Finally, let us prove that the constructed set A is the minimal dimensional support of the measure ν.

Let us consider an arbitrary support C of the measure ν, that is, $\nu(C)=1$. It is clear that the set $C_1=C\cap A$ is also a support of the measure ν and that $C_1\subset C$. Then $\dim_H(C_1)\leq\dim_H(C)$ and $C_1\subset A$. Let us prove that $\dim_H(C_1)=\dim_H(A)$.

From $C_1\subset A$ it follows that $\dim_H(C_1)\leq\dim_H(A)=D$. On the other hand,

$$C_1\subset A\subset A_3=\left\{x:\quad \underline{\lim}_{n\to\infty}\frac{\ln\nu(\Delta_n(x))}{\ln\mu(\Delta_n(x))}\geq D\right\}.$$

Therefore, from the Billingsley theorem [8] it follows that

$$\dim_H(C_1)=\dim_\mu(C_1,\Phi)\geq D\cdot\dim_\nu(C_1,\Phi)=D\cdot 1=D.$$

So, $\dim_H(C_1) = D = \dim_H(A)$.

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