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ANTISYMMETRIC ARENS REGULARITY

TARAS VASYLYSHYN AND YURII SHARYN

Abstract. We introduce a concept of antisymmetric Arens regularity of Banach spaces and prove that a Banach space is Arens regular if and only if it is both symmetrically Arens regular and antisymmetrically Arens regular. We show that if a Banach space is nonreflexive, then it is not antisymmetrically Arens regular and provide an example of a symmetrically Arens regular Banach space which is not antisymmetrically Arens regular.

1. Introduction

It is known that every continuous bilinear map on the Cartesian product of Banach spaces $X \times Y$ with values in a Banach space Z can be extended to a continuous bilinear map on $X'' \times Y''$ with values in Z'' using the so-called Arens extension. Here X'' is the second dual of X. In the general case, the Arens extension of a given bilinear map is not unique. A Banach space is Arens regular (the precise definitions are in the main part of the paper) if the Arens extension of any continuous bilinear map on $X \times X$ is unique. The concept of Arens regularity is important in the theory Banach algebras and infinite-dimensional complex analysis. In [3], it was considered the property of symmetric Arens regularity of a Banach space X, which provides that every symmetric bilinear map on $X \times X$ has a unique Arens extension. In particular, it was shown that the spectrum of the algebra of entire functions of bounded type on X (that is, bounded on bounded subsets of X) admits an analytic structure over X'' if and only if X is symmetrically Arens regular. More about algebraic and analytic structures on the spectrum of the algebra of entire functions of bounded type can be found in [2, 5, 6, 9] and in references therein.

In this note, we introduce a concept of antisymmetric Arens regularity of Banach spaces and prove that a Banach space is Arens regular if and only if it is both symmetrically Arens regular and antisymmetrically Arens regular. Also, we show that if X is nonreflexive, then it is not antisymmetrically Arens regular.

2. Preliminaries

Arens Regularity. Let us recall some basic definitions and known results.

Definition 2.1. Let X and Y be Banach spaces. A bilinear operator $m : X \times X \to Y$ is said to be **symmetric** (respectively, **antisymmetric**), if the following condition

$$m(x, y) = m(y, x) \qquad \big(\text{respectively, } m(x, y) = -m(y, x)\big)$$

holds for all $x, y \in X$.

We use the definition of an adjoint of a bilinear operator presented in [1].

Definition 2.2 ([1])**.** Let X, Y, Z be Banach spaces. The adjoint operator $m^* : Z' \times X \to Y'$ of a bilinear operator $m : X \times Y \to Z$ is defined for $f \in Z'$ and $x \in X$ by

$$m^*(f, x)(y) = f(m(x, y)), \qquad y \in Y.$$

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Note: when reading this paper, it is helpful to keep in mind that

$$\begin{aligned} &m : X \times Y \to Z, \\ &m^* : Z' \times X \to Y', \\ &m^{**} : Y'' \times Z' \to X', \\ &m^{***} : X'' \times Y'' \to Z''. \end{aligned}$$

If $m : X \times Y \to Z$ is a bilinear operator, we can define the transposed operator $m^t : Y \times X \to Z$ by

$$m^t(y, x) = m(x, y).$$

Since m^{t***} is an extension of m^t, the operator m^{t***t} is another extension of m. The following question naturally arises: is it true that $m^{t***t} = m^{***}$? If so, we say that m is *regular*.

Definition 2.3. Let X be a real Banach space.

- X is called *Arens regular* (or simply *regular*) if every bilinear form $m : X \times X \to \mathbb{R}$ is regular.
- X is called *symmetrically regular* if every symmetric bilinear form $m : X \times X \to \mathbb{R}$ is regular.

We will also use the following definition.

Definition 2.4. ([3]) Let X be a real Banach space. Let $m : X \times X \to \mathbb{R}$ be a bilinear form. The linear mapping *associated* with m is defined by $T : x \in X \mapsto m(x, \cdot) \in X'$.

Quasi-reflexive Banach space. Let X be a Banach space. Let $\pi : X \to X''$ be the canonical embedding. The space X is called quasi-reflexive of order n if the factor space $X''/\pi(X)$ is n-dimensional. Let us construct an example of such a space.

James type space ([7, Section 1]).

Let $\mathcal{E} = \{e_i\}_{i=1}^{\infty}$ be a normalized basis of a Banach space X. Let $\|\cdot\|_X$ be the norm of X. Denote by c_{00} the space of all finitely supported scalar sequences. Let

$$u_i = (\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots)$$

for $i \in \mathbb{N}$. Define the following norm on c_{00}. For $(a_1, a_2, \dots) \in c_{00}$, let

$$\left\| \sum_{i=1}^{\infty} a_i u_i \right\|_{c_{00},\mathcal{E}} = \sup \left\{ \left\| \sum_{i=1}^{k} \left(\sum_{j=p(i)}^{q(i)} a_j \right) e_{p(i)} \right\|_X : \ k \in \mathbb{N}, \right.$$
$$\left. p(1), q(1), \dots, p(k), q(k) \in \mathbb{N},\ p(1) \le q(1) < \dots < p(k) \le q(k) \right\}.$$

The completion of c_{00} with respect to this norm is denoted by $J(\mathcal{E})$. Let us assume that X is reflexive. In this case, by a combination of [4, Theorem 2.2] and [4, Theorem 4.1], we see that $J(\mathcal{E})$ is quasi-reflexive of order one. In the following, we will use the next result from [7].

Theorem 2.5 ([7, Theorem 12])**.** *Let $\mathcal{E} = \{e_i\}_{i=1}^{\infty}$ be the unit vector basis of ℓ_p for some p with $2 \le p < \infty$. Then $J(\mathcal{E})'$ is symmetrically regular but not regular.*

3. Antisymmetric Arens Regularity

In this section, we introduce the notion of *antisymmetric Arens regularity* for Banach spaces. We also present an example of a space that is symmetrically regular but not antisymmetrically regular. The main objective of this section is to prove Theorem 3.4 below.

Definition 3.1. Let X be a Banach space. We say that X is *antisymmetrically regular* if every antisymmetric bilinear form $m : X \times X \to \mathbb{R}$ is regular.

Theorem 3.2. *A Banach space X is Arens regular if and only if it is both symmetrically and antisymmetrically regular.*

Proof. Necessity. Suppose that X is Arens regular. By definition, this means that every bilinear form $m : X \times X \to \mathbb{R}$ is regular. In particular, this applies to both symmetric and antisymmetric bilinear forms. Hence, X is symmetrically as well as antisymmetrically Arens regular.

Sufficiency. Assume that X is both symmetrically and antisymmetrically regular. That is, every symmetric and every antisymmetric bilinear form is regular. It is well known that any bilinear form m admits the decomposition $m = m_s + m_a$, where

$$m_s(x,y) = \tfrac{1}{2}\big(m(x,y) + m(y,x)\big), \qquad m_a(x,y) = \tfrac{1}{2}\big(m(x,y) - m(y,x)\big).$$

By assumption, both m_s and m_a are regular. Consequently,

$$m^{***} = m_s^{***} + m_a^{***} = m_s^{t***t} + m_a^{t***t} = m^{t***t}.$$

Thus m itself is regular. Since this holds for every bilinear form, X is Arens regular. $\square$

Corollary 3.3. *The Banach space $J(\mathcal{E})'$, constructed on the unit vector basis of ℓ_p, where $p \in [2, +\infty)$, is symmetrically regular but not antisymmetrically regular.*

Proof. From Theorem 2.5 we have that $J(\mathcal{E})'$ is symmetrically regular but not regular. Thus, by Theorem 3.2, $J(\mathcal{E})'$ must be not antisymmetrically regular. $\square$

In [3], it is proved that if X is nonreflexive, then $X \times X'$ is not symmetrically regular. The following theorem shows that the same is true for the antisymmetric regularity.

Theorem 3.4. *Let X be a nonreflexive Banach space. Then $X \times X'$ is not antisymmetrically regular.*

To prove Theorem 3.4, it is necessary to extend the symmetric bilinear form m_s to m_s^{***}, since this extension will be used to establish the loss of antisymmetry of the operator m_{as} under its extension to m_{as}^{***}, thereby showing that $X \times X'$ is not antisymmetrically regular.

Proposition 3.5. *Let X be a nonreflexive Banach space. Consider the symmetric bilinear functional*

$$m_s : \big(X \times X'\big) \times \big(X \times X'\big) \to \mathbb{R}, \qquad m_s\big((x,f),(y,g)\big) = g(x) + f(y).$$

Then the third adjoint $m_s^{***} : (X'' \times X''') \times (X'' \times X''') \to \mathbb{R}$ has the form

$$m_s^{***}\big((x'',f''),(y'',g'')\big)(c) = c\,x''(g'' \circ \pi) + c\,f''(y''), \tag{1}$$

where $\pi : X \to X''$ is the canonical embedding.

Moreover

$$\begin{aligned} m_s^{***}\big((x'',f''),(y'',g'')\big)(c) - m_s^{***}\big((y'',g''),(x'',f'')\big)(c) \\ = c\big(x''(g'' \circ \pi) + f''(y'') - y''(f'' \circ \pi) - g''(x'')\big). \end{aligned} \tag{2}$$

Proof. Step 1. Extension to m_s^*. By Definition 2.2, the adjoint operator

$$m_s^* : \mathbb{R} \times (X \times X') \to X' \times X''$$

is defined for $c \in \mathbb{R}$ and $(x,f) \in X \times X'$ by

$$m_s^*\big(c,(x,f)\big)(y,g) = c \cdot m_s\big((x,f),(y,g)\big) = cg(x) + cf(y), \quad (y,g) \in X \times X'.$$

In other words, $m_s^*\big(c,(x,f)\big)$ corresponds to the pair $(cp_f, c\omega_x) \in X' \times X''$, where

$$p_f(y) := f(y), \qquad \omega_x(g) := g(x). \tag{3}$$

Thus $p_f = f \in X'$ and $\omega_x = \pi(x) \in X''$, where $\pi : X \to X''$ is the canonical embedding.

Step 2. Extension to m_s^{**}. Next, extend to

$$m_s^{**} : (X'' \times X''') \times \mathbb{R} \to X' \times X'',$$

defined for $(y'', g'') \in X'' \times X'''$, $c \in \mathbb{R}$ by

$$\begin{aligned} m_s^{**}\big((y'', g''), c\big)(x, f) &= (y'', g'')\big(m_s^*(c, (x, f))\big) \\ &= y''(cf) + g''(c\pi(x)), \quad (x, f) \in X \times X'. \end{aligned}$$

Hence

$$m_s^{**}\big((y'', g''), c\big)(x, f) = cy''(f) + cg''(\pi(x)).$$

Equivalently, this action corresponds to $(cq_{g''}, c\delta_{y''})$, where

$$q_{g''}(x) = g''(\pi(x)), \qquad \delta_{y''}(f) = y''(f).$$

Step 3. Extension to m_s^{***}. Finally,

$$m_s^{***} : (X'' \times X''') \times (X'' \times X''') \to \mathbb{R}$$

is defined by

$$m_s^{***}\big((x'', f''), (y'', g'')\big)(c) = (x'', f'')\big(m_s^{**}\big((y'', g''), c\big)\big), \quad c \in \mathbb{R}.$$

Expanding, we get

$$\begin{aligned} (x'', f'')\big(m_s^{**}((y'', g''), c)\big) &= (x'', f'')(cq_{g''}, c\delta_{y''}) \\ &= x''(cq_{g''}) + f''(c\delta_{y''}) \\ &= c\,x''(g'' \circ \pi) + c\,f''(y''). \end{aligned}$$

Therefore,

$$m_s^{***}\big((x'', f''), (y'', g'')\big)(c) = c\,x''(g'' \circ \pi) + c\,f''(y'').$$

Swapping the arguments gives

$$m_s^{***}\big((y'', g''), (x'', f'')\big)(c) = c\,y''(f'' \circ \pi) + c\,g''(x'').$$

Subtracting yields the antisymmetric part:

$$\begin{aligned} &m_s^{***}\big((x'', f''), (y'', g'')\big)(c) - m_s^{***}\big((y'', g''), (x'', f'')\big)(c) \\ &\qquad = c\big(x''(g'' \circ \pi) + f''(y'') - y''(f'' \circ \pi) - g''(x'')\big). \end{aligned}$$

□

Proposition 3.6. *Let X be a non-reflexive Banach space. Consider the antisymmetric bilinear form*

$$m_{as} : \big(X \times X'\big) \times \big(X \times X'\big) \to \mathbb{R}, \qquad m_{as}\big((x, f), (y, g)\big) = g(x) - f(y).$$

Then the third adjoint

$$m_{as}^{***} : (X'' \times X''') \times (X'' \times X''') \to \mathbb{R}$$

has the form

$$m_{as}^{***}\big((x'', f''), (y'', g'')\big)(c) = c\,x''(g'' \circ \pi) - c\,f''(y''). \tag{4}$$

where $\pi : X \to X''$ is the canonical embedding.

Moreover

$$\begin{aligned} &m_{as}^{***}\big((x'', f''), (y'', g'')\big)(c) + m_{as}^{***}\big((y'', g''), (x'', f'')\big)(c) \\ &\qquad = c\big(x''(g'' \circ \pi) - f''(y'') + y''(f'' \circ \pi) - g''(x'')\big), \end{aligned} \tag{5}$$

Proof. Extension to m^*. According to Definition 2.2, the first adjoint operator

$$m_{as}^* : \mathbb{R} \times (X \times X') \to X' \times X''$$

is defined for $c \in \mathbb{R}$ and $(x, f) \in X \times X'$ by

$$m_{as}^*\big(c, (x, f)\big)(y, g) = c \cdot m_{as}\big((x, f), (y, g)\big) = cg(x) - cf(y), \quad (y, g) \in X \times X'.$$

In other words, the action of $m_{as}^*\big(c, (x, f)\big)$ on $(y, g) \in X \times X'$ is given by the pair $(-cp_f, c\omega_x) \in X' \times X''$, where functionals p_f and ω_x are defined by (3).

Extension to m_{as}^{**}. We now extend the operator m_{as}^* to

$$m_{as}^{**} : (X'' \times X''') \times \mathbb{R} \to X' \times X'',$$

which is defined for $(y'', g'') \in X'' \times X'''$ and $c \in \mathbb{R}$ by

$$\begin{aligned} m_{as}^{**}\big((y'', g''), c\big)(x, f) &= \big(y'', g''\big)\big(m_{as}^*\big(c, (x, f)\big)\big) \\ &= y''(-cf) + g''(c\pi(x)) \\ &= cg''(\pi(x)) - cy''(f), \qquad (x, f) \in X \times X'. \end{aligned}$$

We see, that this action can be described by the pair $(cq_{g''}, -c\delta_{y''})$ of functionals, where

$$q_{g''}(x) = g''(\pi(x)), \qquad \delta_{y''}(f) = y''(f).$$

Extension to m_{as}^{***}. Finally, we define the third adjoint

$$m_{as}^{***} : (X'' \times X''') \times (X'' \times X''') \to \mathbb{R}$$

for $(x'', f'') \in X'' \times X'''$ and $(y'', g'') \in X'' \times X'''$ by

$$m_{as}^{***}\big((x'', f''), (y'', g'')\big)(c) = \big(x'', f''\big)\big(m_{as}^{**}\big((y'', g''), c\big)\big), \qquad c \in \mathbb{R}.$$

Expanding the right-hand side of the above equation, we obtain

$$\begin{aligned} \big(x'', f''\big)\big(m_{as}^{**}\big((y'', g''), c\big)\big) &= (x'', f'')(cq_{g''}, -c\delta_{y''}) \\ &= x''(cq_{g''}) - f''(c\delta_{y''}) = c\, x''(g'' \circ \pi) - c\, f''(y''). \end{aligned}$$

Therefore,

$$m_{as}^{***}\big((x'', f''), (y'', g'')\big)(c) = c\, x''(g'' \circ \pi) - c\, f''(y'')$$

is explicit formula for m_{as}^{***}.

From the equation (4) we get

$$m_{as}^{***}\big((y'', g''), (x'', f'')\big)(c) = c\, y''(f'' \circ \pi) - c\, g''(x'').$$

To obtain symmetric part of extended form we will construct the following sum

$$\begin{aligned} m_{as}^{***}&\big((x'', f''), (y'', g'')\big)(c) + m_{as}^{***}\big((y'', g''), (x'', f'')\big)(c) \\ &= c\big(x''(g'' \circ \pi) - f''(y'') + y''(f'' \circ \pi) - g''(x'')\big). \end{aligned}$$

□

Proof of Theorem 3.4. Define the symmetric bilinear operator

$$m_s((x, f), (y, g)) := g(x) + f(y),$$

considered in Proposition 3.5, which shows us explicit formula (1) for m_s^{***} and antisymmetric part (2) of m_s^{***}. If m_s were regular, (2) would be zero for all $(x'', f'') \in X'' \times X'''$ and $(y'', g'') \in X'' \times X'''$. However, as noted in the proof of [3, Corollary 1.8], the associated with m_s linear mapping $T : (x, f) \in X \times X' \mapsto (f, x) \in (X \times X')'$, is not weakly compact when X is non-reflexive. Consequently, by [8, Definition 1.1], m_s is not weakly compact. Therefore, by [8, Theorem 2.2], m_s is not regular. Consequently, there exist some $(x'', f'') \in X'' \times X'''$ and $(y'', g'') \in X'' \times X'''$ such that (2) is nonzero.

Now consider the antisymmetric bilinear operator

$$m_{as}\big((x,f),(y,g)\big) := g(x) - f(y).$$

From the computations in Proposition 3.6, we obtain that the antisymmetry condition for m_{as}^{***} is equivalent to the right-hand side of equation (5) being equal to zero. Take the quadruple $(x'', -y'', f'', g'')$, where $(x'', f'') \in X'' \times X'''$ and $(y'', g'') \in X'' \times X'''$ are as above, making (2) nonzero. Substituting $-y''$ for y'' in the right-hand side of (5) gives

$$\begin{aligned} c\big(x''(g''\circ\pi) &- f''(-y'') + (-y'')(f''\circ\pi) - g''(x'')\big) \\ &= c\big(x''(g''\circ\pi) + f''(y'') - y''(f''\circ\pi) - g''(x'')\big), \end{aligned}$$

which is exactly the right-hand side of (2). Since this value is nonzero by the choice of (x'', y'', f'', g''), the condition (5) fails, and m_{as}^{***} is not antisymmetric.

Thus, there is a bilinear operator on $X \times X'$ whose third adjoint does not preserve antisymmetry, so $X \times X'$ is not antisymmetrically regular. □

As we mentioned above, the James type Banach space $J(\mathcal{E})'$ is symmetrically regular but not antisymmetrically regular. We do not know whether an antisymmetrically regular Banach space is symmetrically regular in the general case. In [3, Proposition 1.9], it is proved that if $X \times X$ is symmetrically regular, then X is regular. In other words, if X is isomorphic to its Cartesian square $X \times X$, then X is symmetrically regular if and only if it is regular. From here and Theorem 3.2 we have that if X is isomorphic to $X \times X$, then X is symmetrically regular if and only if it is antisymmetrically regular. Thus, nontrivial examples of antisymmetrically regular Banach spaces can exist only among spaces that are not isomorphic to their Cartesian square. It is well known that James type Banach spaces are not isomorphic to their Cartesian square.

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Taras Vasylyshyn: taras.vasylyshyn@cnu.edu.ua

Department of Mathematics and Computer Science, Vasyl Stefanyk Carpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine

Yurii Sharyn: yurii.sharyn.21@pnu.edu.ua
Department of Mathematics and Computer Science, Vasyl Stefanyk Carpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine

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