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TWO-FERMION LATTICE HAMILTONIAN WITH NEAREST-NEIGHBOR AND NEXT-TO-NEAREST-NEIGHBOR INTERACTIONS

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Abstract. We study the Schrödinger operators $H_{\lambda\mu}(K)$, with $K \in \mathbb{T}^2$ the fixed quasimomentum of the particles pair, of a system of two identical fermions on the two-dimensional lattice $\mathbb{Z}^2$ with nearest-neighbor and next-to-nearest-neighbor interactions $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, respectively. We find certain conditions for the parameters λ, $\mu \in \mathbb{R}$ of the potential, which independent of the quasimomentum $K \in \mathbb{T}^2$, such that for all $K \in \mathbb{T}^2$ the lattice Schrödinger operator $H_{\lambda\mu}(K)$ has six eigenvalues lying only below or only above the essential spectrum.

1. Introduction

Lattice models play a significant role in various fields of physics. These models include the lattice two-body and three-body Hamiltonians, which can be seen as a simplified version of the Bose- or Fermi-Hubbard models, involving a fixed number of particles of a specific type. It is worth noting that these two and three body lattice Hamiltonians have significant theoretical interest on their own [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].In addition, these discrete Hamiltonians may be viewed as a natural approximation for their continuous counterparts [11] allowing to study few-body phenomena in the context of the theory of bounded operators.

Ultracold atoms trapped in periodic potentials provide a natural physical realization of lattice two-body Hamiltonians, and several experimental works have demonstrated the formation of bound pairs–both attractively bound and repulsively bound dimers–in such systems. Notable examples include the observation of repulsively bound atom pairs in optical lattices [12] and fermionic molecular pairing near a Feshbach resonance [13, 14]. These developments, together with comprehensive reviews of optical-lattice physics [15], provide strong motivation for studying discrete Schrödinger operators with finite-range interactions as mathematically tractable and physically relevant models for fermionic dimers in optical lattices.

Unlike in the continuous case, the lattice two-body system does not admit separation of the center-of-mass motion. However, the discrete translation invariance allows one to use the Floquet-Bloch-type decomposition (see, e.g., [16, Section 4]). In particular, the total two-particle lattice Hamiltonian $\mathbb{H}$ of a system of two identical fermions on the two-dimensional lattice $\mathbb{Z}^2$ with short range interactions, in the (quasi)momentum representation can be decomposed as the von Neumann direct integral

$$\mathbb{H} \simeq \int\limits_{K\in\mathbb{T}^2}^{\oplus} H(K)\, dK, \tag{1.1}$$

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where $\mathbb{T}^2$ is the two-dimensional torus and $H(K)$, the fiber Hamiltonian acting in the respective functional Hilbert space on $\mathbb{T}^2$. Recall that the fiber Hamiltonians $H(K)$ nontrivially depend on the quasimomentum $K \in \mathbb{T}^2$ (see e.g., [16, 17, 18]).

In order to receive a more detail information, we consider an interaction between particles that contains only two terms, the one which is non-trivial if the particles are located in the nearest neighboring sites of the lattice, and another one, non-trivial if the particles are situated in the next to nearest neighbouring sites (see Section 2.1, definition (2.1)). These terms include real factors (coupling constants) λ and μ, respectively, and, in the (quasi)momentum representation, the combined interaction operator is denoted by $V_{\lambda\mu}$.

Thus, as the entries $H(K)$ in (1.1), in this paper, we consider the family of the fiber operators

$$H_{\lambda\mu}(K) := H_0(K) + V_{\lambda\mu}, \qquad K = (K_1, K_2) \in \mathbb{T}^2, \tag{1.2}$$

where $H_0(K)$ is the fiber kinetic-energy operator acting as a multiplication operator in Hilbert space $L^{2,o}(\mathbb{T}^2)$.

The interaction $V_{\lambda\mu}$ is an integral operator in Hilbert space $L^{2,o}(\mathbb{T}^2)$ with a smooth kernel function explicitly given by formula (2.5) below. The formula (2.5) implies that for any non-zero λ, μ the operator $V_{\lambda\mu}$ is of rank 6. Notice that $V_{\lambda\mu}$ does not depend on K at all. The operators $H_0(K)$ and $V_{\lambda\mu}$, $\lambda, \mu \in \mathbb{R}$, are bounded and self-adjoint. Since $V_{\lambda\mu}$ is finite rank, the essential spectrum of $H_{\lambda\mu}(K)$ coincides with the spectrum of $H_0(K)$ (see Section 2.4), i.e. it coincides with the segment $[\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$, where

$$\mathcal{E}_{\min}(K) := 2\sum_{i=1}^{2}\left(1 - \cos\tfrac{K_i}{2}\right), \qquad \mathcal{E}_{\max}(K) := 2\sum_{i=1}^{2}\left(1 + \cos\tfrac{K_i}{2}\right).$$

To the best of our knowledge, the Hamiltonian (1.2) represents a new exactly solvable model [19]. In this model, we prove that there are certain conditions on the parameters λ and μ, such that under these conditions, the lattice Schrödinger operator $H_{\lambda\mu}(K)$ for all $K \in \mathbb{T}^2$ has the maximum number of eigenvalues below or above the bottom of its essential spectrum. (See Theorem 3.2). More precisely, the main result is determined by the explicit constant $\mu_1^- < 0$. Accordingly, the threshold values for the interaction parameters are given by $\mu_1^- - 8$ and $-\mu_1^- + 8$. We prove that if

$$\lambda < \mu_1^- - 8 \quad \text{and} \quad \mu < \mu_1^- - 8,$$

then for every $K \in \mathbb{T}^2$ the operator $H_{\lambda\mu}(K)$ has exactly six eigenvalues located strictly below the essential spectrum and no eigenvalues above it. Conversely, if

$$\lambda > -\mu_1^- + 8 \quad \text{and} \quad \mu > -\mu_1^- + 8,$$

then $H_{\lambda\mu}(K)$ has exactly six eigenvalues located strictly above the essential spectrum and no eigenvalues below it.

To achieve this goal, we use as a technical tool, the Fredholm determinants [20, 21]. Namely, we consider the Fredholm determinant $\Delta_{\lambda\mu}(K, z)$ associated with the Lippmann-Schwinger operator generated by the unperturbed Hamiltonian $H_0(K)$ and perturbation operator $V_{\lambda\mu}$. It is well known [20, 21] that for any fixed $K \in \mathbb{T}^2$, there is a one-to-one mapping between the set of eigenvalues of the Schrödinger operator $H_{\lambda\mu}(K) = H_0(K) + V_{\lambda\mu}$ and the set of zeros of the associated Fredholm determinant $\Delta_{\lambda\mu}(K, z)$.

We start with a study of the properties of the determinant $\Delta_{\lambda\mu}(z) := \Delta_{\lambda\mu}(0, z)$ in the edge case $K = 0$. Assuming that the function $C^-(\lambda, \mu)$ (respectively $C^+(\lambda, \mu)$) is the main (constant) term of the asymptotics of the determinant $\Delta_{\lambda\mu}(z)$ as z approaches to the lower (respectively upper) threshold of the essential spectrum, we show that an additional root of $\Delta_{\lambda\mu}(z)$ emerges below (respectively above) the essential spectrum of $H_{\lambda\mu}(0)$ if $C^-(\lambda, \mu) = 0$ (respectively $C^+(\lambda, \mu) = 0$) (See [19], Theorem 3.2).

In papers [10, 22, 23], the isolated eigenvalues of the two-particle Schrödinger operator $H_{\mu,\lambda}(K)$ for $K \in \mathbb{T}^2$, was studied. This operator is associated with the Bose-Hubbard model for a system of two identical bosons interacting on site and nearest-neighbour sites in the $d \geq 1$ dimensional lattice $\mathbb{Z}^d$, with interaction strengths $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, respectively. It was established that the plane of interaction parameters (λ, μ) is partitioned so that in each connected component of this plane, the Schrödinger operator $H_{\lambda\mu}(0)$ has a fixed number of eigenvalues. These eigenvalues can be either below or above the essential spectrum. Moreover, it has been found that, in each such connected component, the number of isolated eigenvalues of the operator $H_{\lambda\mu}(0)$ is determined and a sharp lower bound is given for the number of eigenvalues of the Schrödinger operator $H(K)$ for all values of quasimomentum $K \in \mathbb{T}^d$.

In [24], analogous results were obtained for Schrödinger operators $H_{\gamma\lambda\mu}(K)$ associated with a system of two identical bosons on the one-dimensional lattice $\mathbb{Z}$, where the real parameters γ, λ, and μ describe interactions between pairs of particles on site, the two nearest neighboring sites, and the next two neighboring sites, respectively.

The family $H_{a,b}(\mu) = H_0 + \mu V_{a,b}$ of Schrödinger-type operators on the two dimensional lattice $\mathbb{Z}^2$, where H_0 is a Laurent-Toeplitz-type convolution operator with a given Hopping matrix ε and $V_{a,b}$ is a potential taking into account only the zero-range and one-range interactions have been considered in [25]. The discrete spectrum of $H_{a,b}(\mu)$, which lies outside the essential spectrum, was completely described under certain conditions on the regularity of the function ε and depending on parameters μ, a and b. Moreover, the threshold eigenfunctions and resonances have been characterized.

It is well known that one-dimensional continuous Schrödinger operators with compactly supported negative potentials may have exactly one, two, or any prescribed finite number of negative eigenvalues; see, for example, Landau–Lifshitz [26], Teschl [27], and Reed–Simon [28].

The paper is organized as follows. Section 1 is introduction. In Section 2 we introduce the two-particle Hamiltonian in the position and quasimomentum representations. Decomposing the Hamiltonian $\mathbb{H}_{\lambda\mu}$ into the von Neumann direct integral we find the lattice Schrödinger operator in the corresponding functional Hilbert space. Section 3 contains statements of our main results and their proofs.

2. Two-fermion lattice Hamiltonian

2.1. **The two-fermion lattice Hamiltonian in the position–space representation.** Let $\mathbb{Z}^2$ be the two-dimensional lattice and $\ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2) \subset \ell^2(\mathbb{Z}^2 \times \mathbb{Z}^2)$, the Hilbert space of square–summable antisymmetric functions on $\mathbb{Z}^2 \times \mathbb{Z}^2$.

In the position-space representation, the Hamiltonian $\widehat{\mathbb{H}}_{\lambda\mu}$ associated with a system of two fermions with a first and second nearest-neighboring-site interaction potential $\widehat{\mathbb{V}}_{\lambda\mu}$ is an operator on $\ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2)$ of the following form:

$$\widehat{\mathbb{H}}_{\lambda\mu} = \widehat{\mathbb{H}}_0 + \widehat{\mathbb{V}}_{\lambda\mu}, \ \ \lambda, \mu \in \mathbb{R}. \tag{2.3}$$

Here, $\widehat{\mathbb{H}}_0$ is the kinetic energy operator of the system, defined on $\ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2)$ as

$$[\widehat{\mathbb{H}}_0 \hat{f}](x_1, x_2) = \sum_{s_1 \in \mathbb{Z}^2} \hat{\varepsilon}(x_1 - s_1)\hat{f}(s_1, x_2) + \sum_{s_2 \in \mathbb{Z}^2} \hat{\varepsilon}(x_2 - s_2)\hat{f}(x_1, s_2), \ \ \hat{f} \in \ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2),$$

where

$$\hat{\varepsilon}(s) = \begin{cases} 2, & |s| = 0, \\ -\frac{1}{2}, & |s| = 1, \\ 0, & |s| > 1, \end{cases}$$

with $|s| = |s_1| + |s_2|$ for $s = (s_1, s_2) \in \mathbb{Z}^2$. The first and second nearest-neighboring-site interaction potential $\widehat{\mathbb{V}}_{\lambda\mu}$ is the operator of multiplication by a function $\hat{v}_{\lambda\mu}$,

$$[\widehat{\mathbb{V}}_{\lambda\mu}\hat{f}](x_1, x_2) = \hat{v}_{\lambda\mu}(x_1 - x_2)\hat{f}(x_1, x_2), \ \ \hat{f} \in \ell^{2,a}(\mathbb{Z}^2 \times \mathbb{Z}^2),$$

where

$$\hat{v}_{\lambda\mu}(s) = \begin{cases} \frac{\lambda}{2}, & |s| = 1, \\ \frac{\mu}{2}, & |s| = 2, \\ 0, & otherwise. \end{cases}$$

Recall that all the three operators $\widehat{\mathbb{H}}_0$, $\widehat{\mathbb{V}}_{\lambda\mu}$, and $\widehat{\mathbb{H}}_{\lambda\mu}$ (for $\lambda, \mu \in \mathbb{R}$) are bounded and self-adjoint.

2.2. **The two-fermion lattice Hamiltonian in the quasimomentum space representation.** Let $\mathbb{T}^2$ be the two-dimensional torus, $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2 \equiv [-\pi, \pi)^2$. The torus $\mathbb{T}^2$ represents the Pontryagin dual group of $\mathbb{Z}^2$, equipped with the Haar measure $\mathrm{d}p$. Let $L^{2,a}(\mathbb{T}^2 \times \mathbb{T}^2)$ be the Hilbert space of square-integrable antisymmetric functions on $\mathbb{T}^2 \times \mathbb{T}^2$.

The quasimomentum-space version of the Hamiltonian (2.3) reads as

$$\mathbb{H}_{\lambda\mu} := (\mathcal{F} \otimes \mathcal{F})\hat{\mathbb{H}}_{\lambda\mu}(\mathcal{F} \otimes \mathcal{F})^*,$$

where $\mathcal{F} \otimes \mathcal{F}$ denotes the Fourier transform. The operator $\mathbb{H}_{\lambda\mu}$ acts on $L^{2,a}(\mathbb{T}^2 \times \mathbb{T}^2)$ and has the form $\mathbb{H}_{\lambda\mu} = \mathbb{H}_0 + \mathbb{V}_{\lambda\mu}$, where $\mathbb{H}_0 = (\mathcal{F} \otimes \mathcal{F})\hat{\mathbb{H}}_0(\mathcal{F} \otimes \mathcal{F})^*$ is the multiplication operator:

$$[\mathbb{H}_0 f](p, q) = [\varepsilon(p) + \varepsilon(q)]f(p, q),$$

with

$$\varepsilon(p) := \sum_{i=1}^{2} \big(1 - \cos p_i\big), \quad p = (p_1, p_2) \in \mathbb{T}^2,$$

being the *dispersion relation* of a single fermion. The interaction $\mathbb{V}_{\lambda\mu} = (\mathcal{F} \otimes \mathcal{F})\hat{\mathbb{V}}_{\lambda\mu}(\mathcal{F} \otimes \mathcal{F})^*$ is the integral operator

$$[\mathbb{V}_{\lambda\mu} f](p, q) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} v_{\lambda\mu}(p - u) f(u, p + q - u)\mathrm{d}u$$

with the kernel function

$$v_{\lambda\mu}(p) = \lambda \sum_{i=1}^{2} \cos p_i + \mu \sum_{i=1}^{2} \cos 2p_i + \mu \sum_{i=1}^{2} \sum_{i \neq j = 1}^{2} \cos p_i \cos p_j, \quad p = (p_1, p_2) \in \mathbb{T}^2.$$

2.3. **The Floquet-Bloch-type decomposition of $\mathbb{H}_{\lambda\mu}$ and lattice Schrödinger operators.** Since $\hat{\mathbb{H}}_{\lambda\mu}$ commutes with the representation of the discrete group $\mathbb{Z}^2$ by shift operators on the lattice, the space $L^{2,a}(\mathbb{T}^2 \times \mathbb{T}^2)$ and $\mathbb{H}_{\lambda\mu}$ can be decomposed into the von Neumann direct integral as (see, e.g., [16])

$$L^{2,a}(\mathbb{T}^2 \times \mathbb{T}^2) \simeq \int_{K \in \mathbb{T}^2}^{\oplus} L^{2,o}(\mathbb{T}^2)\, \mathrm{d}K$$

and

$$\mathbb{H}_{\lambda\mu} \simeq \int_{K \in \mathbb{T}^2}^{\oplus} H_{\lambda\mu}(K)\, \mathrm{d}K,$$

where $L^{2,o}(\mathbb{T}^2)$ is the Hilbert space of odd functions on $\mathbb{T}^2$. The fiber operator $H_{\lambda\mu}(K)$, $K \in \mathbb{T}^2$, in (2.3) acting on $L^{2,o}(\mathbb{T}^2)$ is of the form

$$H_{\lambda\mu}(K) := H_0(K) + V_{\lambda\mu},$$

where the (unperturbed) operator $H_0(K)$ is the multiplication operator by the function

$$\mathcal{E}_K(p) := 2\sum_{i=1}^{2}\Big(1-\cos\tfrac{K_i}{2}\,\cos p_i\Big), \qquad p\in\mathbb{T}^2, \tag{2.4}$$

and the perturbation operator $V_{\lambda\mu}$ is given by

$$\begin{aligned}[V_{\lambda\mu}f](s) =& \frac{\lambda}{(2\pi)^2}\sum_{i=1}^{2}\sin s_i\int\limits_{\mathbb{T}^2}\sin t_i f(t)\mathrm{d}t+\frac{\mu}{(2\pi)^2}\sum_{i=1}^{2}\sin 2s_i\int\limits_{\mathbb{T}^2}\sin 2t_i f(t)\mathrm{d}t \\ &+\frac{\mu}{(2\pi)^2}\sum_{i=1}^{2}\sum_{i\neq j=1}^{2}\sin s_i\cos s_j\int\limits_{\mathbb{T}^2}\sin t_i\cos t_j f(t)\mathrm{d}t.\end{aligned} \tag{2.5}$$

Note that both operators $H_0(K)$ and $V_{\lambda\mu}$ are bounded and self-adjoint. In the literature, the parameter $K\in\mathbb{T}^2$ is called the *two-particle quasimomentum* and the fiber $H_{\lambda\mu}(K)$ is called the *discrete Schrödinger operator* associated to the two-particle Hamiltonian $\widehat{\mathbb{H}}_{\lambda\mu}$.

2.4. **The essential spectrum of lattice Schrödinger operators.** Depending on $\lambda,\mu\in\mathbb{R}$, the rank of $V_{\lambda\mu}$ varies but never exceeds six. Hence, by Weyl's theorem, for any $K\in\mathbb{T}^2$ the essential spectrum $\sigma_{\rm ess}(H_{\lambda\mu}(K))$ of $H_{\lambda\mu}(K)$ coincides with the spectrum of $H_0(K)$, i.e.,

$$\sigma_{\rm ess}(H_{\lambda\mu}(K))=\sigma(H_0(K))=[\mathcal{E}_{\min}(K),\mathcal{E}_{\max}(K)],$$

with

$$\mathcal{E}_{\min}(K):=\min_{p\in\mathbb{T}^2}\mathcal{E}_K(p)=2\sum_{i=1}^{2}\Big(1-\cos\tfrac{K_i}{2}\Big)\geq\mathcal{E}_{\min}(0)=0,$$

$$\mathcal{E}_{\max}(K):=\max_{p\in\mathbb{T}^2}\mathcal{E}_K(p)=2\sum_{i=1}^{2}\Big(1+\cos\tfrac{K_i}{2}\Big)\leq\mathcal{E}_{\max}(0)=8,$$

where $\mathcal{E}_K$ is given by (2.4).

3. Main results and proofs

The following result is an analogue of Theorems 1 and 2 in [16].

Theorem 3.1. *Suppose that $H_{\lambda\mu}(0)$ has n eigenvalues below and above the essential spectrum for some $\lambda,\mu\in\mathbb{R}$. Then for each $K\in\mathbb{T}^2$ the operator $H_{\lambda\mu}(K)$ has at least n eigenvalues below and above its essential spectrum, respectively.*

Theorem 3.1 is proved in [19]. □

Denote by $\mu_0^{\pm}$ and $\mu_1^{\pm}$ the following numbers:

$$\mu_0^{\pm}=\frac{88-30\pi\pm\sqrt{1044\pi^2-6720\pi+10816}}{240\pi-24\pi^2-512}\pi, \tag{3.6}$$

and

$$\mu_1^{\pm}=\frac{128-16\pi-9\pi^2\pm\sqrt{225\pi^4-1440\pi^3+3904\pi^2-10240\pi+16384}}{120\pi-12\pi^2-256}. \tag{3.7}$$

Note that the numerical values of $\mu_0^{\pm}$ and $\mu_1^{\pm}$ are as follows:

$$\mu_0^-=-5.6172...,\quad \mu_0^+=-2.0623...,\quad \mu_1^-=-5.7523...,\quad \mu_1^+=-2.9272...,$$

and, hence, these numbers satisfy the relations

$$\mu_1^-<\mu_0^-<\mu_1^+<\mu_0^+<0. \tag{3.8}$$

Our main results are given in the following theorem.

Theorem 3.2. *Let $K \in \mathbb{T}^2$ and $(\lambda, \mu) \in \mathbb{R}^2$.*

(i) *If $\lambda < \mu_1^- - 8$, $\mu < \mu_1^- - 8$ then $H_{\lambda\mu}(K)$ has six eigenvalues below the essential spectrum and has no eigenvalues above the essential spectrum.*
(ii) *If $\lambda > -\mu_1^- + 8$, $\mu > -\mu_1^- + 8$ then $H_{\lambda\mu}(K)$ has six eigenvalues above the essential spectrum and has no eigenvalues below the essential spectrum.*

We postpone the proof of Theorem 3.2 after several Lemmas.

Let $L^{2,\mathrm{os}}(\mathbb{T}^2) \subset L^{2,\mathrm{o}}(\mathbb{T}^2)$ and $L^{2,\mathrm{oa}}(\mathbb{T}^2) \subset L^{2,\mathrm{o}}(\mathbb{T}^2)$ be the subspaces of odd-symmetric and odd-antisymmetric functions defined as

$$L^{2,\mathrm{os}}(\mathbb{T}^2) = \{f \in L^{2,o}(\mathbb{T}^2) : f(p_1, p_2) = f(p_2, p_1), \text{for a.e } (p_1, p_2) \in \mathbb{T}^2\}$$

and

$$L^{2,\mathrm{oa}}(\mathbb{T}^2) = \{f \in L^{2,o}(\mathbb{T}^2) : f(p_1, p_2) = -f(p_2, p_1), \text{for a.e } (p_1, p_2) \in \mathbb{T}^2\},$$

respectively.

Lemma 3.3. *The equality*

$$L^{2,\mathrm{o}}(\mathbb{T}^2) = L^{2,\mathrm{os}}(\mathbb{T}^2) \oplus L^{2,\mathrm{oa}}(\mathbb{T}^2)$$

holds true.

Proof. Let $f \in L^{2,\mathrm{o}}(\mathbb{T}^2)$. Define the following two functions:

$$f_{\mathrm{os}}(p_1, p_2) = \frac{1}{2}\big[f(p_1, p_2) + f(p_2, p_1)\big],$$
$$f_{\mathrm{oa}}(p_1, p_2) = \frac{1}{2}\big[f(p_1, p_2) - f(p_2, p_1)\big].$$

By direct calculation, we have $f = f_{\mathrm{os}} + f_{\mathrm{oa}}$ with $f_{\mathrm{os}} \in L^{2,\mathrm{os}}(\mathbb{T}^2)$ and $f_{\mathrm{oa}} \in L^{2,\mathrm{oa}}(\mathbb{T}^2)$.

Moreover, $f_{\mathrm{os}} \perp f_{\mathrm{oa}}$. Indeed, set $h(p_1, p_2) := f_{\mathrm{os}}(p_1, p_2)\overline{f_{\mathrm{oa}}(p_1, p_2)}$. Since $f_{\mathrm{os}}(p_2, p_1) = f_{\mathrm{os}}(p_1, p_2)$ and $f_{\mathrm{oa}}(p_2, p_1) = -f_{\mathrm{oa}}(p_1, p_2)$, we get $h(p_2, p_1) = -h(p_1, p_2)$. Hence, by the change of variables $(p_1, p_2) \mapsto (p_2, p_1)$ (which preserves the Haar measure on $\mathbb{T}^2$),

$$(f_{\mathrm{os}}, f_{\mathrm{oa}}) = \int_{\mathbb{T}^2} h(p_1, p_2)\, \mathrm{d}p_1 \mathrm{d}p_2 = \int_{\mathbb{T}^2} h(p_2, p_1)\, \mathrm{d}p_1 \mathrm{d}p_2 = -\int_{\mathbb{T}^2} h(p_1, p_2)\, \mathrm{d}p_1 \mathrm{d}p_2 = 0.$$

□

The operator $H_0(0)$ is the multiplication operator by the symmetric function $\mathcal{E}_0(p) = 2\varepsilon(p) = 2\varepsilon(p_1, p_2)$ in $L^{2,o}(\mathbb{T}^2)$. Hence, for each $\theta \in \{\mathrm{os}, \mathrm{oa}\}$ the subspace $L^{2,\theta}(\mathbb{T}^2)$ is invariant with respect to $H_0(0)$. Now recall that the interaction operator $V_{\lambda\mu}$ has the form (2.5) and, thus, it reads as

$$\begin{aligned}[V_{\lambda\mu} f](p) =& \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} (\sin p_1 \sin q_1 + \sin p_2 \sin q_2) f(q) \mathrm{d}q \\ &+ \frac{\mu}{(2\pi)^2} \int_{\mathbb{T}^2} (\sin 2p_1 \sin 2q_1 + \sin 2p_2 \sin 2q_2) f(q) \mathrm{d}q \\ &+ \frac{\mu}{2\pi^2} \int_{\mathbb{T}^2} (\sin p_1 \cos p_2 \sin q_1 \cos q_2 + \sin p_2 \cos p_1 \sin q_2 \cos q_1) f(q) \mathrm{d}q.\end{aligned}$$

By applying the equalities

$$\begin{aligned}
&2\sin p_1\sin q_1+2\sin p_2\sin q_2\\
&\qquad=(\sin p_1+\sin p_2)(\sin q_1+\sin q_2)+(\sin p_1-\sin p_2)(\sin q_1-\sin q_2),\\
&2\sin 2p_1\sin 2q_1+2\sin 2p_2\sin 2q_2\\
&\qquad=(\sin 2p_1+\sin 2p_2)(\sin 2q_1+\sin 2q_2)\\
&\qquad+(\sin 2p_1-\sin 2p_2)(\sin 2q_1-\sin 2q_2),\\
&2\sin p_1\cos p_2\sin q_1\cos q_2+2\sin p_2\cos p_1\sin q_2\cos q_1\\
&\qquad=(\sin p_1\cos p_2+\sin p_2\cos p_1)(\sin q_1\cos q_2+\sin q_2\cos q_1)\\
&\qquad+(\sin p_1\cos p_2-\sin p_2\cos p_1)(\sin q_1\cos q_2-\sin q_2\cos q_1)
\end{aligned}$$

and denoting by $V^{\theta}_{\lambda\mu}$ the respective part of $V_{\lambda\mu}$ in the reducing subspace $L^{2,\theta}(\mathbb{T}^2)$, $\theta\in\{\mathrm{os},\mathrm{oa}\}$, we arrive at the expressions

$$\begin{aligned}
[V^{\mathrm{os}}_{\lambda\mu}f](p)=&\frac{\lambda}{8\pi^2}(\sin p_1+\sin p_2)\int\limits_{\mathbb{T}^2}(\sin q_1+\sin q_2)f(q)\mathrm{d}q\\
&+\frac{\mu}{8\pi^2}(\sin 2p_1+\sin 2p_2)\int\limits_{\mathbb{T}^2}(\sin 2q_1+\sin 2q_2)f(q)\mathrm{d}q\\
&+\frac{\mu}{4\pi^2}(\sin p_1\cos p_2+\sin p_2\cos p_1)\int\limits_{\mathbb{T}^2}(\sin q_1\cos q_2+\sin q_2\cos q_1)f(q)\mathrm{d}q
\end{aligned}$$

and

$$\begin{aligned}
[V^{\mathrm{oa}}_{\lambda\mu}f](p)=&\frac{\lambda}{8\pi^2}(\sin p_1-\sin p_2)\int\limits_{\mathbb{T}^2}(\sin q_1-\sin q_2)f(q)\mathrm{d}q\\
&+\frac{\mu}{8\pi^2}(\sin 2p_1-\sin 2p_2)\int\limits_{\mathbb{T}^2}(\sin 2q_1-\sin 2q_2)f(q)\mathrm{d}q\\
&+\frac{\mu}{4\pi^2}(\sin p_1\cos p_2-\sin p_2\cos p_1)\int\limits_{\mathbb{T}^2}(\sin q_1\cos q_2-\sin q_2\cos q_1)f(q)\mathrm{d}q.
\end{aligned}$$

It follows from the above expressions that the subspaces $L^{2,\mathrm{os}}(\mathbb{T}^2)$ and $L^{2,\mathrm{oa}}(\mathbb{T}^2)$ are invariant with respect to $V_{\lambda\mu}$. Since they are also invariant with respect to $H_0(0)$, they are invariant with respect to $H_{\lambda\mu}(0)$.

Consequently,

$$H_{\lambda\mu}(0)\big|_{L^{2,\theta}(\mathbb{T}^2)}=H^{\theta}_{\lambda\mu}(0):=H_0+V^{\theta}_{\lambda\mu},\ \text{for}\ \theta\in\{\mathrm{os},\mathrm{oa}\}.$$

Therefore,

$$\sigma\Big(H_{\lambda\mu}(0)\Big)=\bigcup_{\theta\in\{\mathrm{os},\mathrm{oa}\}}\sigma\Big(H_{\lambda\mu}(0)\big|_{L^{2,\theta}(\mathbb{T}^2)}\Big),$$

where $A\big|_{\mathcal{H}}$ is the restriction of a self-adjoint operator A on a reducing subspace $\mathcal{H}$. Thus, the study of the discrete spectrum of $H_{\lambda\mu}(0)$ reduces to that for the restrictions of $H_{\lambda\mu}(0)$ onto each subspace $L^{2,\theta}(\mathbb{T}^2)$, $\theta\in\{\mathrm{os},\mathrm{oa}\}$.

3.1. **The Lippmann–Schwinger operator.** Let $\{\alpha^{\theta}_i,\ i=1,2,3\}$ be a system of vectors in $L^{2,\theta}(\mathbb{T}^2)$, $\theta\in\{\mathrm{os},\mathrm{oa}\}$, with

$$\alpha^{os}_1(p)=\frac{\sin p_1+\sin p_2}{2\pi},\ \alpha^{os}_2(p)=\frac{\sin 2p_1+\sin 2p_2}{2\pi},\ \alpha^{os}_3(p)=\frac{\sin p_1\cos p_2+\sin p_2\cos p_1}{\sqrt{2}\pi}\tag{3.9}$$

and

$$\alpha_1^{oa}(p) = \frac{\sin p_1 - \sin p_2}{2\pi},\ \alpha_2^{oa}(p) = \frac{\sin 2p_1 - \sin 2p_2}{2\pi},\ \alpha_3^{oa}(p) = \frac{\sin p_1 \cos p_2 - \sin p_2 \cos p_1}{\sqrt{2}\pi} \tag{3.10}$$

One easily verifies by inspection that the vectors (3.9) and (3.10) are orthonormal in $L^{2,o}(\mathbb{T}^2)$. By using the orthonormal systems (3.9) and (3.10) one obtains

$$V_{\lambda\mu}^{\theta} f = \frac{\lambda}{2}(f, \alpha_1^{\theta})\alpha_1^{\theta} + \frac{\mu}{2}(f, \alpha_2^{\theta})\alpha_2^{\theta} + \frac{\mu}{2}(f, \alpha_3^{\theta})\alpha_3^{\theta}, \quad \theta \in \{\text{os},\ \text{oa}\}, \tag{3.11}$$

where $(\cdot, \cdot)$ is the inner product in $L^{2,\theta}(\mathbb{T}^2)$.

Definition 3.4 (Transpose Lippmann–Schwinger operator)**.** For $z \in \mathbb{C} \setminus [0, 8]$, the *transpose Lippmann–Schwinger operator* is defined on the Hilbert space $L^{2,\theta}(\mathbb{T}^2)$, $\theta \in \{\text{os}, \text{oa}\}$, by

$$B_{\lambda\mu}^{\theta}(0, z) := -V_{\lambda\mu}^{\theta} R_0(0, z),$$

where

$$R_0(0, z) := (H_0(0) - zI)^{-1}, \qquad z \in \mathbb{C} \setminus [0, 8],$$

is the resolvent of the free lattice Schrödinger operator $H_0(0)$(cf. the classical Lippmann-Schwinger construction [29]). The terminology "transpose" reflects that the interaction operator $V_{\lambda\mu}^{\theta}$ acts on the left of the resolvent, in contrast to the conventional Lippmann–Schwinger form $R_0 V$. The notation $B_{\lambda\mu}^{\theta}(0, z)$ indicates the dependence on the fiber quasimomentum fixed at $K = 0$ and on the spectral parameter z.

Note that for each $z \in \mathbb{C} \setminus [0, 8]$ the resolvent $R_0(0, z) = (H_0(0) - zI)^{-1}$ is a bounded operator on $L^{2,\theta}(\mathbb{T}^2)$. Since $V_{\lambda\mu}^{\theta}$ is a finite-rank operator, $B_{\lambda\mu}^{\theta}(0, z)$ is compact in $L^{2,\theta}(\mathbb{T}^2)$.

Lemma 3.5. *For each $\lambda, \mu \in \mathbb{R}$ the number $z \in \mathbb{C} \setminus [0, 8]$ is an eigenvalue of the operator $H_{\lambda\mu}^{\theta}(0)$ if and only if the number 1 is an eigenvalue for $B_{\lambda\mu}^{\theta}(0, z)$, $\theta \in \{\text{os}, \text{oa}\}$.*

Proof. The identity

$$\left(I - B_{\lambda\mu}^{\theta}(0, z)\right)(H_0(0) - zI) = H_{\lambda\mu}^{\theta}(0) - zI \tag{3.12}$$

follows directly from Definition 3.4. Since $H_0(0) - zI$ is invertible for $z \in \mathbb{C} \setminus [0, 8]$, the operator $H_{\lambda\mu}^{\theta}(0) - zI$ is not invertible if and only if $I - B_{\lambda\mu}^{\theta}(0, z)$ is not invertible. Equivalently, $1 \in \sigma\left(B_{\lambda\mu}^{\theta}(0, z)\right)$. This yields a lattice analogue of the classical Birman–Schwinger principle (cf. [20, Theorem 1.3.1]). □

By Lemma 3.5, the eigenvalue problem

$$H_{\lambda\mu}^{\theta}(0)\varphi = z\varphi$$

is equivalent to the Lippmann–Schwinger equation

$$B_{\lambda\mu}^{\theta}(0, z)\varphi = \varphi, \qquad \varphi \in L^{2,\theta}(\mathbb{T}^2). \tag{3.13}$$

The representation (3.11) yields the equivalence of the Lippmann-Schwinger equation (3.13) to the following algebraic linear system in $x_i := (\varphi, \alpha_i^{\theta})$, $i = 1, 2, 3$:

$$\begin{cases} (1 + \lambda a^{\theta}(z))x_1 + \lambda c^{\theta}(z)x_2 + \lambda d^{\theta}(z)x_3 = 0, \\ \mu c^{\theta}(z)x_1 + (1 + \mu b^{\theta}(z))x_2 + \mu e^{\theta}(z)x_3 = 0, \\ 2\mu d^{\theta}(z)x_1 + 2\mu e^{\theta}(z)x_2 + (1 + \mu f^{\theta}(z))x_3 = 0, \end{cases} \tag{3.14}$$

where

$$a^\theta(z) = \frac{1}{8\pi^2}\int\limits_{\mathbb{T}^2} \frac{(\sin p_1 \pm \sin p_2)^2 \ \mathrm{d}p}{\mathcal{E}_0(p) - z}, \tag{3.15}$$

$$b^\theta(z) = \frac{1}{8\pi^2}\int\limits_{\mathbb{T}^2} \frac{(\sin 2p_1 \pm \sin 2p_2)^2 \ \mathrm{d}p}{\mathcal{E}_0(p) - z}, \tag{3.16}$$

$$c^\theta(z) = \frac{1}{8\pi^2}\int\limits_{\mathbb{T}^2} \frac{(\sin p_1 \pm \sin p_2)(\sin 2p_1 \pm \sin 2p_2)\mathrm{d}p}{\mathcal{E}_0(p) - z}, \tag{3.17}$$

$$d^\theta(z) = \frac{\sqrt{2}}{8\pi^2}\int\limits_{\mathbb{T}^2} \frac{(\sin p_1 \pm \sin p_2)(\sin p_1 \cos p_2 \pm \sin p_2 \cos p_1)\mathrm{d}p}{\mathcal{E}_0(p) - z}, \tag{3.18}$$

$$e^\theta(z) = \frac{\sqrt{2}}{8\pi^2}\int\limits_{\mathbb{T}^2} \frac{(\sin 2p_1 \pm \sin 2p_2)(\sin p_1 \cos p_2 \pm \sin p_2 \cos p_1)\mathrm{d}p}{\mathcal{E}_0(p) - z}, \tag{3.19}$$

$$f^\theta(z) = \frac{1}{2\pi^2}\int\limits_{\mathbb{T}^2} \frac{(\sin p_1 \cos p_2 \pm \sin p_2 \cos p_1)^2\mathrm{d}p}{\mathcal{E}_0(p) - z}. \tag{3.20}$$

Here the sign "$\pm$" corresponds to $\theta \in \{\mathrm{os}, \mathrm{oa}\}$ in the natural way: $+$ for $\theta = \mathrm{os}$ and $-$ for $\theta = \mathrm{oa}$. Since the domain $\mathbb{T}^2$ is symmetric and the measure is invariant under $(p_1, p_2) \mapsto (p_1, -p_2)$, the mixed terms containing $\sin p_1 \sin p_2$, $\sin 2p_1 \sin 2p_2$, etc. are odd in p_2 and therefore integrate to zero. Consequently, each of the integrals in (3.15)–(3.20) is the same for the choices "$+$" and "$-$", that is, $a^{\mathrm{os}}(z) = a^{\mathrm{oa}}(z)$, ..., $f^{\mathrm{os}}(z) = f^{\mathrm{oa}}(z)$. Thus, we omit the superscript θ and write these functions simply as $a(z), b(z), c(z), d(z), e(z)$, and $f(z)$.

The reasoning similar to the one we used in the case of the functions (3.15)–(3.20) allows us to conclude that, although the operator $B^\theta_{\lambda\mu}(0, z)$ depends on θ through its domain $L^{2,\theta}(\mathbb{T}^2)$, the determinant of the operator $I - B^\theta_{\lambda\mu}(0, z)$ does not depend on θ, too. Thus, for any $\lambda, \mu \in \mathbb{R}$ and $z \in \mathbb{C} \setminus [0, 8]$, we define

$$\Delta_{\lambda\mu}(z) := \det[I - B^\theta_{\lambda\mu}(0, z)] = \begin{vmatrix} 1 + \lambda a(z) & \lambda c(z) & \lambda d(z) \\ \mu c(z) & 1 + \mu b(z) & \mu e(z) \\ 2\mu d(z) & 2\mu e(z) & 1 + \mu f(z) \end{vmatrix}. \tag{3.21}$$

Lemma 3.6. *A number $z \in \mathbb{C} \setminus [0, 8]$ is an eigenvalue of the operator $H_{\lambda\mu}(0)$ if and only if*

$$\Delta_{\lambda\mu}(z) = 0.$$

If z is such an eigenvalue then necessarily it has multiplicity two.

Proof. The claim follows from Lemma 3.5 and the explicit determinant representation (3.21). Indeed, 1 is an eigenvalue of $B^\theta_{\lambda\mu}(0, z)$ if $\det[I - B^\theta_{\lambda\mu}(0, z)] = 0$, which by (3.21) gives precisely (3.6). Conversely, if $\Delta_{\lambda\mu}(z_0) = 0$, then there exists a nontrivial $c = (c_1, c_2, c_3)^\top \in \mathbb{C}^3$ solving (3.14). The corresponding eigenfunction of $H^\theta_{\lambda\mu}(0)$ is given by

$$f^\theta(p) = \lambda c_1 \left(R_0(0, z_0)\alpha^\theta_1\right)(p) + \mu c_2 \left(R_0(0, z_0)\alpha^\theta_2\right)(p) + \mu c_3 \left(R_0(0, z_0)\alpha^\theta_3\right)(p),$$

where $\alpha^\theta_1(p), \alpha^\theta_2(p)$ and $\alpha^\theta_3(p)$ are defined in (3.9) and (3.10). Finally, since

$$H_{\lambda\mu}(0) = H^{\mathrm{os}}_{\lambda\mu}(0) \oplus H^{\mathrm{oa}}_{\lambda\mu}(0),$$

the number z_0 is an eigenvalue of multiplicity two for the full operator $H_{\lambda\mu}(0)$. □

Lemma 3.7. *For any $\lambda, \mu \in \mathbb{R}$ the determinant $\Delta_{\lambda\mu}(z)$ has the form*

$$\Delta_{\lambda\mu}(z) = \Delta_{\lambda 0}(z)\Delta_{0\mu}(z) + \Delta^{(12)}_{\lambda\mu}(z), \tag{3.22}$$

where

$$\Delta_{\lambda 0}(z) = 1 + \lambda a(z), \tag{3.23}$$

$$\Delta_{0\mu}(z) = (1 + \mu b(z))(1 + \mu f(z)) - 2\mu^2 e^2(z), \tag{3.24}$$

$$\Delta^{(12)}_{\lambda\mu}(z) = 4\lambda\mu^2 c(z)d(z)e(z) - \lambda\mu c^2(z)(1 + \mu f(z)) - 2\lambda\mu d^2(z)(1 + \mu b(z)).$$

Proof. A direct computation of the determinant gives the result. □

Lemma 3.8. *The functions $a(z), b(z), c(z), d(z), e(z)$ and $f(z)$ defined in $\mathbb{R} \setminus [0, 8]$ are real-valued and, moreover, strictly increasing and positive in $(-\infty, 0)$, strictly increasing and negative in $(8, +\infty)$ and have following asymptotics:*

$$\lim_{z\nearrow 0} a(z) = \frac{\pi - 2}{2\pi}, \quad \lim_{z\nearrow 0} b(z) = \frac{30\pi - 92}{3\pi}, \quad \lim_{z\nearrow 0} c(z) = \frac{2\pi - 6}{\pi},$$
$$\lim_{z\nearrow 0} d(z) = \frac{4 - \pi}{2\sqrt{2}\pi}, \quad \lim_{z\nearrow 0} e(z) = \frac{20 - 6\pi}{3\sqrt{2}\pi}, \quad \lim_{z\nearrow 0} f(z) = \frac{4}{3\pi}$$

and

$$\lim_{z\searrow 8} a(z) = \frac{2 - \pi}{2\pi}, \quad \lim_{z\searrow 8} b(z) = \frac{92 - 30\pi}{3\pi}, \quad \lim_{z\searrow 8} c(z) = \frac{6 - 2\pi}{\pi},$$
$$\lim_{z\searrow 8} d(z) = \frac{\pi - 4}{2\sqrt{2}\pi}, \quad \lim_{z\searrow 8} e(z) = \frac{6\pi - 20}{3\sqrt{2}\pi}, \quad \lim_{z\searrow 8} f(z) = -\frac{4}{3\pi}.$$

Proof. We first consider the case $z < 0$; the case $z > 8$ is treated analogously. The monotonicity of $a(z)$, $b(z)$ and $f(z)$ follows directly from the monotonicity of $(\mathcal{E}_0(p) - z)^{-1}$ with respect to z.

The monotonicity of $c(z)$, $d(z)$ and $e(z)$ follow directly from their integral representations

$$c(z) = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} \frac{\sin^2 p_1 \cos^2 p_1}{A(z, p_2)^2 - \cos^2 p_1} \, dp_1 \right] dp_2,$$

$$d(z) = \frac{\sqrt{2}}{8\pi^2} \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} \frac{\sin^2 p_2 \cos^2 p_1}{A(z, p_2)^2 - \cos^2 p_1} \, dp_1 \right] dp_2,$$

$$e(z) = \frac{\sqrt{2}}{8\pi^2} \int_{0}^{\pi} \left[\int_{0}^{\pi} \frac{64B(z) \sin^2 p_1 \cos^2 p_1 \cos^2 p_2}{(B(z)^2 - 4(\cos p_1 + \cos p_2)^2)(B(z)^2 - 4(\cos p_1 - \cos p_2)^2)} \, dp_1 \right] dp_2,$$

where $A(z, p_2) = 2 - \frac{z}{2} - \cos p_2$ and $B(z) = 4 - z$.

We now prove the asymptotics for $d(z)$ as $z \nearrow 0$. Using symmetry of $\mathcal{E}_0(p)$ under $p_1 \leftrightarrow p_2$ and the cancellation of mixed odd terms, we may write

$$d(z) = \frac{\sqrt{2}}{4\pi^2} \int_{\mathbb{T}^2} \frac{\sin^2 p_1 \, \cos p_2}{\mathcal{E}_0(p) - z} \, dp, \qquad z < 0. \tag{3.25}$$

Fix $p_2 \in \mathbb{T}$ and set $A(z, p_2) > 1$ for $z < 0$. Then

$$\int_{\mathbb{T}} \frac{\sin^2 p_1}{\mathcal{E}_0(p) - z} \, dp_1 = \frac{1}{2} \int_{\mathbb{T}} \frac{\sin^2 p_1}{A(z, p_2) - \cos p_1} \, dp_1,$$

and for $A(z, p_2) > 1$ we use the identity

$$\int_{\mathbb{T}} \frac{\sin^2 p_1}{A(z, p_2) - \cos p_1} \, dp_1 = \pi A(z, p_2) - \pi\sqrt{A(z, p_2)^2 - 1}. \tag{3.26}$$

Substituting (3.26) into (3.25) and passing to the limit $z \nearrow 0$ (by dominated convergence), we obtain

$$\begin{aligned}\lim_{z\nearrow 0} d(z) &= \frac{\sqrt{2}}{8\pi^2}\int_{\mathbb{T}} \cos p_2 \Big(\pi(2-\cos p_2) - \pi\sqrt{(2-\cos p_2)^2-1}\Big)\,dp_2\\ &= \frac{\sqrt{2}}{8\pi}\int_{\mathbb{T}} \Big(-\cos^2 p_2 - \cos p_2\sqrt{(2-\cos p_2)^2-1}\Big)\,dp_2\\ &= \frac{\sqrt{2}}{8\pi}\Big(-\pi - I_d\Big),\end{aligned}$$

where

$$I_d := \int_{\mathbb{T}} \cos p_2\sqrt{(2-\cos p_2)^2-1}\,dp_2.$$

A direct computation (e.g. by the substitution $u = \tan\frac{p_2}{2}$) gives $I_d = -8$, hence

$$\lim_{z\nearrow 0} d(z) = \frac{\sqrt{2}}{8\pi}\big(-\pi+8\big) = \frac{4-\pi}{2\sqrt{2}\pi}.$$

The remaining asymptotics are obtained analogously. □

Lemma 3.9. *The function $\Delta_{\lambda\mu}(z)$ is real-valued on $\mathbb{R}\setminus[0,8]$ and has the following asymptotics:*

$$\Delta_{\lambda\mu}(z) = C^-(\lambda,\mu) + o(1), \ \text{as } z\nearrow 0,$$
$$\Delta_{\lambda\mu}(z) = C^+(\lambda,\mu) + o(1), \ \text{as } z\searrow 8,$$

where, using the numbers $\mu_0^\pm$ and $\mu_1^\pm$ defined in (3.6) and (3.7), respectively, we set

$$C^\pm(\lambda,\mu) := \frac{30\pi - 3\pi^2 - 64}{6\pi^2}\Big[8(\mu\pm\mu_0^+)(\mu\pm\mu_0^-) \mp \lambda(\mu\pm\mu_1^+)(\mu\pm\mu_1^-)\Big]. \tag{3.27}$$

Proof. In view of Lemmas 3.7 and 3.8, a direct inspection of the expression for $\Delta_{\lambda\mu}(z)$, together with the definitions of $\mu_0^\pm$ and $\mu_1^\pm$ in (3.6)–(3.7), yields the claimed asymptotics with $C^\pm(\lambda,\mu)$ given by (3.27). □

Next we study the number and location of roots of the function $\Delta_{0\mu}$ defined in (3.24).

Lemma 3.10. *Let $\mu\in\mathbb{R}$ and let μ_0^- be as in (3.6). Then:*

(i) *For any $\mu<\mu_0^-$ the functions $1+\mu b(z)$ and $1+\mu f(z)$ have unique roots $\eta_1^{(-)}(\mu)$ and $\eta_2^{(-)}(\mu)$ in $(-\infty,0)$, respectively.*

(ii) *For any $\mu>-\mu_0^-$ the functions $1+\mu b(z)$ and $1+\mu f(z)$ have unique roots $\eta_1^{(+)}(\mu)$ and $\eta_2^{(+)}(\mu)$ in $(8,+\infty)$, respectively.*

Proof. We prove the assertion only for the equation $1+\mu b(z)=0$ on $(-\infty,0)$. The remaining statements (for $1+\mu f(z)$, as well as the case $z\in(8,+\infty)$) follow verbatim by the same argument, using the corresponding monotonicity and endpoint limits from Lemma 3.8.

Let $\mu<\mu_0^-$. Recall that $\mu_0^- = -5.6172\ldots$ and

$$\lim_{z\nearrow 0} b(z) = \frac{30\pi-92}{3\pi} = 0.2384\ldots$$

Since $\frac{30\pi-92}{3\pi}>0$, the assumption $\mu<\mu_0^-$ implies

$$\lim_{z\nearrow 0}\big(1+\mu b(z)\big) = 1+\mu\frac{30\pi-92}{3\pi} < 1+\mu_0^-\frac{30\pi-92}{3\pi} = -1.3391\ldots < 0.$$

On the other hand,

$$\lim_{z\to-\infty}\big(1+\mu b(z)\big) = 1.$$

Since the function $z \mapsto 1 + \mu b(z)$ is continuous and strictly decreasing on $(-\infty, 0)$, it follows that the equation

$$1 + \mu b(z) = 0$$

has a unique solution $\eta_1^{(-)}(\mu) \in (-\infty, 0)$. □

Let $\eta_1^{(-)}(\mu)$, $\eta_2^{(-)}(\mu)$ and $\eta_1^{(+)}(\mu)$, $\eta_2^{(+)}(\mu)$ be two different pairs of roots of $\Delta_{0\mu}$ such that $\eta_1^{(-)}(\mu)$, $\eta_2^{(-)}(\mu) \in (-\infty, 0)$ and $\eta_1^{(+)}(\mu), \eta_2^{(+)}(\mu) \in (8, +\infty)$. We set:

$$\eta_{\min}^{-}(\mu) = \min\{\eta_1^{(-)}(\mu), \eta_2^{(-)}(\mu)\},$$
$$\eta_{\max}^{-}(\mu) = \max\{\eta_1^{(-)}(\mu), \eta_2^{(-)}(\mu)\}.$$

and

$$\eta_{\min}^{+}(\mu) = \min\{\eta_1^{(+)}(\mu), \eta_2^{(+)}(\mu)\},$$
$$\eta_{\max}^{+}(\mu) = \max\{\eta_1^{(+)}(\mu), \eta_2^{(+)}(\mu)\}.$$

The next lemma describes the dependence of the number of roots of the function $\Delta_{0\mu}$ in $\mathbb{R} \setminus [0, 8]$ and their location on the value $\mu \in \mathbb{R}$.

Lemma 3.11. *Let $\mu \in \mathbb{R}$ and let μ_0^- be as in (3.6). Then:*

(i) *If $\mu < \mu_0^-$, then $\Delta_{0\mu}(\cdot)$ has exactly two roots $\zeta_1^-(0,\mu)$ and $\zeta_2^-(0,\mu)$. These roots satisfy the inequalities*

$$\zeta_1^-(0,\mu) < \eta_{\min}^-(\mu) \le \eta_{\max}^-(\mu) < \zeta_2^-(0,\mu) < 0.$$

(ii) *If $\mu > -\mu_0^-$, then $\Delta_{0\mu}(\cdot)$ has exactly two roots $\zeta_1^+(0,\mu)$ and $\zeta_2^+(0,\mu)$. These roots satisfy the inequalities*

$$8 < \zeta_2^+(0,\mu) < \eta_{\min}^+(\mu) \le \eta_{\max}^+(\mu) < \zeta_1^+(0,\mu).$$

Proof. Let us prove the item (i). Assume that $\mu < \mu_0^-$. By Lemma 3.9 (since $C^-(0,\mu) > 0$ for $\mu < \mu_0^-$) we have

$$\lim_{z \nearrow 0} \Delta_{0\mu}(z) > 0. \tag{3.28}$$

Moreover, by the dominated convergence theorem,

$$\lim_{z \to -\infty} \Delta_{0\mu}(z) = 1. \tag{3.29}$$

By Lemma 3.10, for any $\mu < \mu_0^-$ the functions $1 + \mu b(z)$ and $1 + \mu f(z)$ have unique zeros, denoted as $\eta_1^{(-)}(\mu)$ and $\eta_2^{(-)}(\mu)$, respectively. Moreover,

$$e^2(\eta_1^{(-)}(\mu)) > 0 \quad \text{and} \quad e^2(\eta_2^{(-)}(\mu)) > 0.$$

In this case, $\Delta_{0\mu}(\cdot)$ takes negative values either at $z_0 = \eta_1^{(-)}(\mu)$ or $z_0 = \eta_2^{(-)}(\mu)$. Using the asymptotics in (3.28) and (3.29) at $z = -\infty$ and at $z = 0$, and the continuity of $\Delta_{0\mu}(\cdot)$, it follows that $\Delta_{0\mu}(\cdot)$ has two zeros:

$$\zeta_1^-(0,\mu) \in (-\infty, z_0) \quad \text{and} \quad \zeta_2^-(0,\mu) \in (z_0, 0).$$

Since $V_{0\mu}^\theta$, $\theta \in \{\mathrm{os}, \mathrm{oa}\}$ is rank-two, from the min-max principle $H_{\lambda\mu}^\theta(0)$ has at most two eigenvalues in $\mathbb{R} \setminus [0, 8]$. By Lemma 3.6 the determinant $\Delta_{0\mu}(z)$ has no more than two roots in $\mathbb{R} \setminus [0, 8]$. Thus, $\Delta_{0\mu}(\cdot)$ has exactly two zeros in the interval $(-\infty, 0)$.

The remaining item is proved in a similar way. □

Proof of Theorem 3.2. (i) Assume that $\lambda < \mu_1^- - 8$, $\mu < \mu_1^- - 8$. First, for each $\theta \in \{\mathrm{os}, \mathrm{oa}\}$, we show that the operator $H_{\lambda\mu}^\theta(0)$ has three eigenvalues below the essential spectrum. On the other hand, since the interaction operator $V_{\lambda\mu}^\theta$ is of rank three, the min–max principle implies that the operator $H_{\lambda\mu}^\theta(0)$ has at most three eigenvalues outside

the essential spectrum. It yields the operator $H^{\theta}_{\lambda\mu}(0)$ does not have any eigenvalues above the essential spectrum.

Since $\mu < 0$, the functions $1 + \mu b(z)$ and $1 + \mu f(z)$ are continuous and monotonously decreasing in $(-\infty, 0)$. Obviously, by Lemma 3.10 (i) we have

$$1 + \mu b(z) > 0,\ 1 + \mu f(z) > 0 \text{ if } z < \eta^{-}_{\min}(\mu),$$
$$1 + \mu b(z) < 0,\ 1 + \mu f(z) < 0 \text{ if } z > \eta^{-}_{\max}(\mu).$$

For the roots $\zeta^{-}_1(0,\mu)$ and $\zeta^{-}_2(0,\mu)$ of the equation $\Delta_{0\mu}(z) = (1 + \mu b(z))(1 + \mu f(z)) - 2\mu^2 e^2(z) = 0$ we then have

$$\sqrt{1 + \mu f(\zeta^{-}_1(0,\mu))}\sqrt{1 + \mu b(\zeta^{-}_1(0,\mu))} = -\sqrt{2}\mu e(\zeta^{-}_1(0,\mu)) \tag{3.30}$$

and

$$\sqrt{-(1 + \mu f(\zeta^{-}_2(0,\mu)))}\sqrt{-(1 + \mu b(\zeta^{-}_2(0,\mu)))} = -\sqrt{2}\mu e(\zeta^{-}_2(0,\mu)).$$

Notice that

$$\begin{aligned} C^{-}(\lambda,\mu) &= \frac{30\pi - 3\pi^2 - 64}{6\pi^2}[8(\mu - \mu^{+}_0)(\mu - \mu^{-}_0) + \lambda(\mu - \mu^{+}_1)(\mu - \mu^{-}_1)] \\ &< \frac{30\pi - 3\pi^2 - 64}{6\pi^2}[8(\mu - \mu^{+}_0)(\mu - \mu^{-}_0) + \lambda(\mu - \mu^{-}_0)(\mu - \mu^{-}_1)] \\ &= \frac{30\pi - 3\pi^2 - 64}{6\pi^2}(\mu - \mu^{-}_0)[(\lambda - \mu^{-}_1 + 8)(\mu - \mu^{-}_1 + 8) \\ &\qquad - 8(\lambda - \mu^{-}_1 + 8) + \mu^{-}_1(\mu - \mu^{-}_1 + 8) - 8\mu^{+}_0] < 0, \end{aligned}$$

where the first inequality follows from

$$\frac{30\pi - 3\pi^2 - 64}{6\pi^2} > 0,$$

relation (3.8), together with the assumptions $\lambda < \mu^{-}_1 - 8$ and $\mu < \mu^{-}_1 - 8$. The last inequality follows after factoring out $(\mu - \mu^{-}_0)$ and using that all remaining factors are negative under the assumptions $\lambda < \mu^{-}_1 - 8$ and $\mu < \mu^{-}_1 - 8$.

Therefore, by Lemma 3.9, it follows that

$$\lim_{z \nearrow 0} \Delta_{\lambda\mu}(z) < 0.$$

Moreover, by the dominated convergence theorem,

$$\lim_{z \to -\infty} \Delta_{\lambda\mu}(z) = 1.$$

Using the explicit representation (3.22) for $\Delta_{\lambda\mu}$ and the identity (3.30) one arrives at the following chain of equalities:

$$\begin{aligned} &\Delta_{\lambda\mu}(\zeta^{-}_1(0,\mu)) \\ &= \Delta_{\lambda 0}(\zeta^{-}_1(0,\mu))\Delta_{0\mu}(\zeta^{-}_1(0,\mu)) + \Delta^{(12)}_{\lambda\mu}(\zeta^{-}_1(0,\mu)) \\ &= 4\lambda\mu^2 c(\zeta^{-}_1(0,\mu))d(\zeta^{-}_1(0,\mu))e(\zeta^{-}_1(0,\mu)) \\ &- \lambda\mu c^2(\zeta^{-}_1(0,\mu))(1 + \mu f(\zeta^{-}_1(0,\mu))) \\ &- 2\lambda\mu d^2(\zeta^{-}_1(0,\mu))(1 + \mu b(\zeta^{-}_1(0,\mu))) \\ &= -\lambda\mu\left(c(\zeta^{-}_1(0,\mu))\sqrt{1 + \mu f(\zeta^{-}_1(0,\mu))} + d(\zeta^{-}_1(0,\mu))\sqrt{2(1 + \mu b(\zeta^{-}_1(0,\mu)))}\right)^2. \end{aligned} \tag{3.31}$$

Clearly, (3.31) implies

$$\Delta_{\lambda\mu}(\zeta^{-}_1(0,\mu)) < 0. \tag{3.32}$$

Analogously, using the identity (3.1) one obtains

$$\Delta_{\lambda\mu}(\zeta_2^-(0,\mu)) > 0. \tag{3.33}$$

Inequalities (3.32) and (3.33) together with the relations

$$\lim_{z\to-\infty}\Delta_{\lambda\mu}(z) = 1, \lim_{z\nearrow 0}\Delta_{\lambda\mu}(z) < 0.$$

yield the existence three roots z_1, z_2 and z_3 of function $\Delta_{\lambda\mu}(z)$.

Hence the function $\Delta_{\lambda\mu}(z)$ has three single roots smaller than 0. From Lemma 3.6 it follows that the operator $H_{\lambda\mu}(0)$ has six eigenvalues (counted with multiplicities) below the essential spectrum. Since the interaction operator $V_{\lambda\mu}$ has rank at most six, the min–max principle implies that $H_{\lambda\mu}(0)$ has at most six eigenvalues in $\mathbb{R}\setminus[0,8]$, and hence $H_{\lambda\mu}(0)$ has no eigenvalues above its essential spectrum. Therefore by using Theorem 3.1 we obtain that the operator $H_{\lambda\mu}(K)$ has six eigenvalue below its essential spectrum. The item (ii) can be proven in a similar way.

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