Methods of Functional Analysis and Topology
Vol. 32 (2026), no. 3, pp. 215–219
doi.org/10.31392/MFAT-npu26_3.2026.02

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

MONOTONICITY PROPERTY OF NON-ARCHIMEDEAN HEAT EQUATION

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Abstract. We prove the monotone dependence of the heat kernel corresponding to the Vladimirov-Taibleson non-Archimedean fractional differentiation operator D^α on the order α.

1. Introduction

Let K be a non-Archimedean local field, $\Omega = \left\{x \in K^n : \ \|x\|_{K^n} \le q^N\right\}$ be an ultrametric ball. Here $N \in \mathbb{Z}$, $\|(x_1, \dots x_n)\|_{K^n} = \max\limits_{1\le j\le n} |x_j|_K$, q is the residue cardinality of K (for the basic notions regarding local fields see Section 2 below and [7, 10, 18]). The best known example of a non-Archimedean local field is the field of p-adic numbers; here p is a prime number.

A model example of a pseudo-differential operator acting on real- or complex-valued functions on Ω is the Vladimirov-Taibleson fractional differential operator

$$\left(D^{\alpha,n}u\right)(x) = c_{n,\alpha} \int\limits_{K^n} \frac{u(x)-u(y)}{\|x-y\|_{K^n}^{n+\alpha}}\, dy, \quad x \in \Omega, \tag{1.1}$$

where dy is the Haar measure, $c_{n,\alpha} = \dfrac{q^\alpha - 1}{1-q^{-\alpha-n}}$, $\alpha > 0$, and a function u is extended onto K^n by a nonlocal Dirichlet condition, that is (for the homogeneous case) $u(y) = 0$, as $y \in \Omega^c = K^n \setminus \Omega$. The precise understanding of the integral in (1.1) depends on smoothness assumptions for u; the simplest case is that of its boundedness and local constancy when $u(y) = u(x)$, as y belongs to a neighbourhood of a point x. Note that K is totally disconnected, and, in contrast to classical analysis, the set of such functions is quite rich.

The choice of the constant $c_{n,\alpha}$ is motivated by the representation of $D^{\alpha,n}$ in terms of the Fourier analysis on K^n; see [10, 17].

The operator (1.1) and related equations have been studied by many authors, see the monographs [17, 10, 1, 16, 9, 19, 20] and many recent papers. Its form, especially its nonlocality, resembles that of the fractional Laplacian $(-\Delta)^{\alpha/2}$ of real analysis, though the actual meaning of the objects of a non-Archimedean theory is quite different.

In this note we consider an analog of the first boundary value problem for the heat equation, that is

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \left(D^{\alpha,n}u\right)(t,x) = 0, \quad x \in \Omega, \ t > 0; \\ u(t,x) = 0, \quad x \in \Omega^c; \\ u(0,x) = \varphi(x), \quad x \in \Omega. \end{cases} \tag{1.2}$$

The non-Archimedean ball Ω is simultaneously open and closed, so that it has no topological boundary. Though one cannot speak, in the classical style, about "boundary value problems", there is a well-developed theory of parabolic equations of this kind. See

2020 *Mathematics Subject Classification.* 11S80; 35S10.
Keywords. Vladimirov-Taibleson operator, local field, pseudo-differential operator, heat kernel.

[10, 2, 19, 20, 11, 3, 4] and references therein. Our aim here is to prove a monotone dependence, in a certain sense, of a solution of (1.2) on α. A similar problem for the fractional Laplacian was studied in [6].

2. Local fields [7, 10, 18]

A (non-Archimedean) local field is a non-discrete totally disconnected locally compact topological field.

An important related notion is that of a field extension. If k is a subfield of a field K, then K is called an algebraic extension of k. In this situation, the field K is a vector space over k, and we call K a finite extension, if this vector space is finite-dimensional. This dimension is called a degree of this extension. A local field K is isomorphic either to a finite algebraic extension of the field $\mathbb{Q}_p$ of p-adic numbers, if K has characteristic 0, or to the field of formal Laurent series with coefficients from a finite field, if K has a positive characteristic. By Ostrowski's theorem, $\mathbb{Q}_p$ is the only possible alternative to $\mathbb{R}$ as a completion of the field of rational numbers. This shows the importance of non-Archimedean mathematics as one of the principal branches of mathematical science.

Any local field is endowed with absolute value $|\cdot|_K$, such that: 1) $|x|_K = 0$, if and only if $x = 0$; 2) $|xy|_K = |x|_K \cdot |y|_K$; 3) $|x + y|_K \le \max(|x|_K, |y|_K)$. The last property is called the ultrametric inequality. It implies the equality $|x + y|_K = |x|_K$, if $|y|_K < |x|_K$.

The ring of integers $O = \{x \in K\colon |x|_K \le 1\}$ contains an ideal $P = \{x \in K\colon |x|_K < 1\}$. There exists such an element β that $P = \beta O$. The quotient ring $\bar{K} = O/P$ is actually a finite field. The absolute value is called normalized, if $|\beta|_K = q^{-1}$, where q is the cardinality of the finite field O/P. Unless stated otherwise, this property is assumed. A normalized absolute value takes exactly the values q^m, $m \in \mathbb{Z}$.

Let L be a finite extension of a local field K. An operator of multiplication in L by an element ξ can be considered as a linear operator on a K-vector space. If the linear function $\xi \mapsto \mathrm{Tr}(\xi)$ does not vanish identically, then the extension is called separable. Here Tr is the trace of a linear operator in a finite-dimensional vector space. All finite extensions of the field of characteristic zero are separable. On the other hand, the notion of separability makes sense also for the finite fields $\bar{K}, \bar{L}$.

A finite extension L of a local field K is called unramified, if $\bar{L}$ is a separable extension of $\bar{K}$ of the same degree, as L over K. Any local field K has a unique (up to the isomorphism) unramified extension of each given degree $n \ge 2$. The cardinality of the residue field $\bar{L}$ in this extension equals q^n where q is the residue cardinality for K.

The unramified extension of degree n has, as a K-vector space, a canonical basis consisting of representatives of a basis in $\bar{L}$ over $\bar{K}$. Let $x \in L$ have the coefficients $x_1, \dots, x_n \in K$ of its expansion with respect to the canonical basis. Then the normalized absolute value $|x|_L$ has the representation [15]:

$$|x|_L = \Big(\max_{1 \le j \le n} |x_j|_K \Big)^n .$$

The comparison of $|x|_L$ with another absolute value on L, $\|x\|_L = |x|_L^{1/n} \equiv \max\limits_{1 \le j \le n} |x_j|_K$, which extends the absolute value from K, shows that the above canonical basis is an orthogonal basis (in the non-Archimedean sense [14]) in L considered as Banach space with the norm $\|\cdot\|_L$. Therefore this basis defines an isometric linear isomorphism $U : L \to K^n$, $Ux = (x_1, \dots, x_n)$.

The above isomorphism identifies the operator $D^{\alpha,n}$ defined by (1.1) with the operator

$$\left(D_L^{\gamma} u\right)(x) = \frac{1 - q^{n\gamma}}{1 - q^{-n(\gamma+1)}} \int\limits_L \frac{u(z) - u(x)}{|z - x|_L^{\gamma+1}}\, dz, \quad \gamma = \frac{\alpha}{n};$$

see [12] for the details.

Thus, a multi-dimentional operator over a local field K is equivalent to a one-dimensional operator over a bigger field, the unramified extension L. Such properties are absent in real analysis (except some two-dimensional operators admitting complex representation).

3. The non-Archimedean heat equation

Let us consider the problem (1.2). We begin with the one-dimensional case ($n = 1$). Using the ultrametric property of the absolute value $|\cdot|_K$, we see that for $x \in \Omega$,

$$\left(D^{\alpha,1}u\right)(t,x) = \left(\Delta_\Omega^\alpha\right)u(t,x) + c_{1,\alpha}u(t,x)\int\limits_{|y|_K>q^N}\frac{dy}{|y|_K^{\alpha+1}} - c_{1,\alpha}\int\limits_{|y|_K>q^N}\frac{u(t,y)dy}{|y|_K^{\alpha+1}} \tag{3.3}$$

where the operator

$$\left(\Delta_\Omega^\alpha\right)u(t,x) = c_{1,\alpha}\int\limits_\Omega \frac{u(t,x)-u(t,y)}{|x-y|_K^{\alpha+1}}\,dy, \quad x \in \Omega,$$

called a local Vladimirov-Taibleson operator in [5], is an analog of the regional fractional operator [8]. The second term on the right in (3.3) equals $\lambda_N^{(\alpha)}u(t,x)$ where

$$\lambda_N^{(\alpha)} = \frac{q-1}{q(1-q^{-\alpha-1})}q^{-\alpha N}$$

(see [11]). The third term on the right in (3.3) equals 0, due to the "boundary condition" in (1.2).

Thus, the problem (1.2) is reduced to a kind of a Cauchy problem (on the open and closed set Ω) for the equation

$$\frac{\partial u(t,x)}{\partial t} + \left(\Delta_\Omega^\alpha u\right)(t,x) + \lambda_N^{(\alpha)}u(t,x) = 0. \tag{3.4}$$

Setting $u(t,x) = e^{-\lambda_N^{(\alpha)}t}v(t,x)$ we come to the equation

$$\frac{\partial v(t,x)}{\partial t} + \left(\Delta_\Omega^\alpha v\right)(t,x) = 0, \quad x \in \Omega, \tag{3.5}$$

the simplest case of a heat equation on a compact p-adic manifold [5].

An orthonormal basis in $L^2(\Omega)$ can be constructed of Kozyrev's p-adic wavelets $\psi : \Omega \to \mathbb{C}$, which are eigenfunctions of Δ_Ω^α corresponding to the eigenvalues [13, 5]

$$\rho_\psi = c_{1,\alpha}\left\{\int\limits_{\Omega\setminus B}|y|_K^{-\alpha-1}dy + (\operatorname{mes} B)^{-\alpha}\right\}, \tag{3.6}$$

where ψ is supported on a subball $B \subset \Omega$. Note that the eigenfunctions ψ do not depend on α.

It is shown in [5] that the heat kernel function (a fundamental solution of the Cauchy problem) corresponding to the equation has the form

$$H_\alpha(t,x,y) = 1 + \sum_\psi e^{-t\rho_\psi}\psi(x)\overline{\psi(y)}, \quad x \neq y \in \Omega, t > 0, \tag{3.7}$$

where ψ runs the set of all the above wavelets. The expansion in (3.7) begins with 1, since the minimal eigenvalue of the operator in equals zero, with the constant eigenfunction.

If $\alpha > 1$, H_α exists also for $x = y$, $t > 0$ and $0 < H(t,\cdot,\cdot) \in L^2(\Omega\times\Omega)$.

Returning to the equation (3.4) we get the corresponding heat kernel

$$G_\alpha(t,x,y) = e^{-\lambda_N^{(\alpha)}t}H_\alpha(t,x,y) \tag{3.8}$$

where H_α is given by (3.7). The above constructions make sense also for the multi-dimensional situation, with an appropriate constant in (3.8).

4. Monotonicity

Now we formulate the main result of this paper.

Theorem 4.1. *Suppose that $N \leq -1$, $n < \alpha_0 < \alpha_1$. Then*

$$G_{\alpha_0}(t,x,x) > G_{\alpha_1}(t,x,x)$$

for all $t > 0$, $x \in \Omega$.

Proof. We begin with the case where $n = 1$. We have

$$\frac{\partial}{\partial\alpha}\lambda_N^{(\alpha)} = -\frac{q-1}{q}q^{-\alpha N}\frac{\log q}{1-q^{-\alpha-1}}\left(\frac{q^{-\alpha-1}}{1-q^{-\alpha-1}}+N\right) > 0,$$

so that $\lambda_N^{(\alpha)}$ increases in α, as $\alpha > 1$. Thus $\exp(-\lambda_N^{(\alpha)}t)$ decreases in α.

Similarly,

$$\frac{\partial}{\partial\alpha}c_{1,\alpha} = \frac{q^\alpha(1-q^{-\alpha-1})\log q - (q^\alpha-1)q^{-\alpha-1}\log q}{(1-q^{-\alpha-1})^2}$$

and

$$q^\alpha(1-q^{-\alpha-1}) - (q^\alpha-1)q^{-\alpha-1} = q^\alpha - 2q^{-1} + q^{-\alpha-1} = q^{-1}(q^{\alpha+1}-2) + q^{-\alpha-1} > 0,$$

since $q \geq 2$, so that $c_{1,\alpha}$ increases in α. It follows from our assumptions regarding Ω and the ultrametric property that the expression in braces in (3.6) increases in α. Thus the eigenvalues ρ_ψ increase, while the expressions $\exp\{-t\rho_\psi\}$ decrease in α, which implies the decrease of the heat kernel.

Turning to the multi-dimensional case, we notice that the isometric isomorphism $U : L \to K^n$ transforms the heat kernel of the operator over K^n into a similar one-dimensional operator over the unramified extension L, so that the multi-dimensional case is a consequence of the one-dimensional one considered for a bigger field. ■

5. Acknowledgements

The authors are grateful to the anonymous referee for helpful remarks. The first author acknowledges the funding support in the framework of the project "Spectral Optimization: From Mathematics to Physics and Advanced Technology" (SOMPATY) received from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 873071. The work of the second author was supported by a grant from the Simons Foundation (1030291,A.N.K.).

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Received 21/04/2026; Revised 10/09/2026