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MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

NEW DEFINITION OF PSEUDO-RESOLVENT OF INTEGRATED SEMI-GROUPS

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Abstract. Using the new definition of the χ_F^α-derivative function introduced by Juan E. Nápoles Valdés and al. (2020), we provide a new definition for the Conformable Semigroup, conformable Laplace transform of semigroup and conformable Integrated semigroup. Additionally, we establish several important results related to these new concepts. This article contributes to the generalization of many previous studies. For example, defining the function F as $F = t^{\alpha-1}$ or $F = 1$ is sufficient to obtain the results obtained.

1. Introduction

Denote by $\rho(\Gamma) = \{\nu \in \mathbb{C} : \nu I - \Gamma \text{ is invertible in } \mathcal{B}(X)\}$ and by

$$\mathcal{R}(;\Gamma) : \rho(\Gamma) \to \mathcal{B}(X)$$
$$\mathcal{R}(,;\Gamma) = (\nu I - \Gamma)^{-1}$$

the resolvent of operator Γ.
We call C_0-semigroup a family $\{\gamma(t)\}_{t\geq 0} \subset B(X)$ verifying the following properties:
i) $\gamma(0) = I$;
ii) $\gamma(t+s) = \gamma(t)\gamma(s), \forall t, s \geq 0$;
iii) $\lim_{t\to 0} \gamma(t)x = x, \forall x \in X$.
And its infinitesimal generator by

$$D(\Gamma) = \{x \in X : \lim_{t\to 0} \frac{\gamma(t)x - x}{t} \text{exists}\}$$

and

$$\Gamma x = \lim_{t\to 0} \frac{\gamma(t)x - x}{t}, \ x \in D(\Gamma)$$

Let:

$$C_{ub}[0,\infty) = \{f : [0,\infty) \to \mathbb{R} : f \text{ is uniformly continuous and bounded}\}$$

With the norm $||f||_{C_{ub}[0,\infty)} = \sup_{\alpha\in[0,\infty)} |f(\alpha)|$, the space $C_{ub}[0,\infty)$ becomes a Banach space. Let us define:

$$(\gamma(t)f)(\alpha) = f(t+\alpha), \forall t \geq 0, \alpha \in [0,\infty)$$

$\{\gamma(t)\}_{t\geq 0}$ is a C_0-semigroup on $C_{ub}[0,\infty)$ and

$$D(\Gamma) = \{f \in C_{ub}[0,\infty) : f' \in C_{ub}[0,\infty)\}$$
$$\Gamma f = f'$$

In this paper, we try to generalize the study on conformable semigroup and conformable integrated semigroups. Where we focus on the pseudo-resolvent.
The structure of this paper is as follows: Definitions and initial findings are given in Section 2, which also reviews the χ_F^α-derivative and χ_F^α-semigroup. The primary findings about the Laplace transform of a semigroup are shown in Section 3. The primary findings pertaining to the conformable integrated semigroup are presented in Section 4.

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2. Basic definitions and tools

Based on the results obtained in Article [2] for definition 1.1, we conclude the following results where we consider: $g(t,\alpha)=\frac{1}{F(t,\alpha)}$ for $F:[0,+\infty[\times(0,1]\to(0,+\infty)$.

Definition 2.1. Let $\alpha\in(0,1]$ and $f:[0,+\infty[\to\mathbb{R}$. Then the N_F^α-derivative of order α is defined by :

$$\chi_F^\alpha(f)(t):=\lim_{h\to 0}\frac{f(t+\frac{h}{F(t,\alpha)})-f(t)}{h}.$$

For all $t\in[0,+\infty[$.

Theorem 2.2. *Let $\alpha\in(0,1]$ and f,g be χ_F^α-differentiable at a point $t>0$. Then:*

(1) $\chi_F^\alpha(af+bg)(t)=a\chi_F^\alpha(f)(t)+b\chi_F^\alpha(g)(t)$.
(2) $\chi_F^\alpha(\lambda)=0,\lambda\in\mathbb{R}$.
(3) $\chi_F^\alpha(fg)(t)=\chi_F^\alpha(f)(t)g(t)+f(t)\chi_F^\alpha(g)(t)$.
(4) $\chi_F^\alpha(\frac{f}{g})(t)=\frac{g(t)\chi_F^\alpha(f)(t)-f(t)\chi_F^\alpha(g)(t)}{g^2(t)}$.
(5) *If, in addition, f is differentiable then* $\chi_F^\alpha(f)(t)=\frac{f'(t)}{F(t,\alpha)}$.

Example 2.3. For $F(t,\alpha)=3\alpha^2t^2+2\alpha^3t+\alpha$, we have

$$\chi_F^\alpha\left(3\alpha^2\frac{t^4}{4}+(2\alpha^3+3\alpha^2)\frac{t^3}{3}+\alpha^3t^2+(\alpha^2+\alpha)t\right)=t^2+1.$$

The author has explained the following properties in Reference [2]:

$$\chi_F^\alpha\{\mathcal{I}_F^\alpha[f](t)\}=f(t)$$

and

$$\mathcal{I}_F^\alpha\{\chi_F^\alpha[f](t)\}=f(t)-f(0)$$

where

$$\mathcal{I}_F^\alpha[f](t)=\int_0^t F(s,\alpha)f(s)ds,\quad t\in[0,+\infty[$$

Example 2.4. For $F(t,\alpha)=\alpha^2t^2+\alpha t,\ t\in[0,+\infty)$ and $\alpha\in(0,1]$, we have

$$\mathcal{I}_F^\alpha(t^2+1)=\alpha^2\frac{t^5}{5}+\alpha\frac{t^4}{4}+\alpha^2\frac{t^3}{3}+\alpha\frac{t^2}{2}.$$

Example 2.5. For $F(t,\alpha)=\alpha^2t^2+\alpha t$, we have by Example 2.4 and Theorem 2.2 that

$$\chi_F^\alpha(\mathcal{I}_F^\alpha(t^2+1))=\frac{\left(\alpha^2\frac{t^5}{5}+\alpha\frac{t^4}{4}+\alpha^2\frac{t^3}{3}+\alpha\frac{t^2}{2}\right)'}{\alpha^2t^2+\alpha t}=t^2+1,$$

for all $t>0$.

Example 2.6. In the Example 2.4, $f(t)=t^2+1$. By Theorem 2.2

$$\mathcal{I}_F^\alpha(\chi_F^\alpha f(t))=\mathcal{I}_F^\alpha\left(\frac{2t}{\alpha^2t^2+\alpha t}\right)=\int_0^t 2sds=t^2=f(t)-f(0),$$

Definition 2.7. [1] Let F and α be as discussed earlier and a Banch space X. Let $G_\alpha(t)=\int_0^t F(\tau,\alpha)d\tau$ be a primitive of F. A family $\{\gamma(t),t\geq 0\}\subset B(X)$ is called a fractional χ_F^α-semigroup of operators if:

(1) $\gamma(0)=I$,
(2) $\gamma(G_\alpha^{-1}(t+s))=\gamma(G_\alpha^{-1}(t))\gamma(G_\alpha^{-1}(s))$, for all $t,s\in[0,\infty)$.

Clearly, if $G_\alpha^{-1}=I$, the χ_F^α-semigroups are just the usual semigroups.

Example 2.8. Let A be a bounded linear operator. Define:

$$G_\alpha(t) = \alpha t^2, t \geq 0$$

and

$$\gamma(t) = e^{\alpha A t^2}$$

$G_\alpha^{-1}(t) = \sqrt{\frac{t}{\alpha}}$. Then

$$\gamma(\sqrt{\frac{t+s}{\alpha}}) = e^{A(t+s)} = e^{At}e^{As} = \gamma(\sqrt{\frac{t}{\alpha}}).\gamma(\sqrt{\frac{s}{\alpha}}).$$

Definition 2.9. An $\mathcal{N}_F^\alpha$-semigroup $\{\gamma(t)\}_{t\geq 0}$ is called $C_0\chi_F^\alpha-$semigroup if, for each fixed $x \in X$, $\gamma(t)x \longrightarrow x \quad$ as $t \longrightarrow 0^+$.

The χ_F^α-derivative of $\gamma(t)$ at $t = 0$ is called the $\chi_F^\alpha-$ infinitesimal generator of the fractional χ_F^α-semigroup $\gamma(t)$, with domain equals

$$\left\{ x \in X : \lim_{t\to 0^+} \chi_F^\alpha \gamma(t)x \text{ exists } \right\}.$$

We will write Γ for such a generator.

In reference [1], the authors proved the following properties:

$$\begin{cases} \chi_F^\alpha \gamma(t)x = \Gamma\gamma(t)x = \gamma(t)\Gamma x. \\ \lim_{h\mapsto 0} \frac{1}{h}\int_t^{t+h} \gamma(u)F(u,\alpha)du = \gamma(t)F(t,\alpha) \quad \forall t \geq 0. \\ \Gamma\left(\int_0^t \gamma(s)F(s,\alpha)x\,\mathrm{d}s\right) = \gamma(t)F(t,\alpha)x - F(0,\alpha)x. \end{cases} \tag{2.1}$$

3. Conformable Laplace transform of Semi-group

Definition 3.1. $\{\gamma(t)\}_{t\geq 0}$ is $\chi_F^\alpha-$exponentially bounded semi-group, if $\exists \omega \geq 0$, $M \geq 1$:

$$||\gamma(t)|| \leq Me^{\omega G_\alpha(t)}.$$

For the rest of this paper, $F > 0$, $\alpha \in (0, 1]$. Let $G_\alpha(t) = \int_0^t F(\tau,\alpha)d\tau$ be a primitive of F and $\{\gamma(t)\}_{t\geq 0}$ is $\chi_F^\alpha-$exponentially bounded semi-group.

Example 3.2. Let us consider $G_\alpha(t) = \alpha sh(t)$ and $\gamma(t) = e^{(\alpha-1)\alpha sh(t)}$.

$$||\gamma(t)|| \leq |e^{-\alpha sh(t)}||e^{\alpha^2 sh(t)}| \leq e^{\alpha G_\alpha(t)}$$

Then $\{\gamma(t)\}_{t\geq 0}$ is $\chi_F^\alpha-$exponentially bounded semi-group. With $M = 1$ and $\omega = \alpha > 0$.

Remark 3.3. There exist integrated semi-groups that are not exponential bounded ([3], example 1.2). For an exponentially bounded integrated semigroup $\{S(t)\}_{t\geq 0}$, the integral $\int_0^{+\infty} e^{-\lambda G_\alpha(t)}\gamma(t)F(t,\alpha)dt$ exists for all $\lambda \in \{\lambda \in \mathbb{C} : Re(\lambda) > \omega\}$ and

$$\left\| \int_0^{+\infty} e^{-\lambda G_\alpha(t)}\gamma(t)F(t,\alpha)dt \right\| \leq \frac{M}{Re(\lambda)-\omega}$$

(Theorem 3.6).

Definition 3.4. Let $\lambda \in \{\lambda \in \mathbb{C} : Re(\lambda) > \omega\}$ and $\{\gamma(t)\}_{t\geq 0}$ be a $\chi_F^\alpha-$exponentially bounded semi-group.

$$\Re_\lambda^c(\gamma(t)) := \Re^c((\gamma(t))(\lambda) = \int_0^{+\infty} e^{-\lambda G_\alpha(t)}\gamma(t)F(t,\alpha)dt$$

is called the Laplace transform of the semi-group, and its inverse is defined by

$$\gamma(t) = \frac{1}{2\pi i} \lim_{\omega \to +\infty} \int_{a-i\omega}^{a+i\omega} e^{\lambda G_\alpha(t)} \Re_\lambda^c(\gamma(t)) F(t,\alpha) d\lambda$$

Example 3.5. Let us consider $G_\alpha(t) = \alpha sh(t)$ and $\gamma(t) = e^{(\alpha-1)\alpha sh(t)}$. Then for all $\lambda \in \mathbb{C} : Re(\lambda) > \alpha$

$$\Re_\lambda^c(\gamma(t)) = \int_0^{+\infty} e^{-\lambda G_\alpha(t)} \gamma(t) F(t,\alpha) dt = \int_0^{+\infty} e^{-(\lambda-\alpha+1)G_\alpha(t)} F(t,\alpha) dt = \frac{1}{\lambda-\alpha+1}.$$

Theorem 3.6. *For all $\lambda \in \{\lambda \in \mathbb{C} : Re(\lambda) > \omega\}$ and $\{\gamma(t)\}_{t\geq 0}$ is an $\chi_F^\alpha-$exponentially bounded semi-group. The operator $\Re_\lambda^c$ is bounded.*

Proof. We have

$$\begin{aligned} ||R_\lambda^c(\gamma(t))|| &\leq \int_0^{+\infty} ||e^{-\lambda G_\alpha(t)} \gamma(t) F(t,\alpha)|| dt \\ &\leq M \int_0^{+\infty} ||e^{-(\lambda-\omega)G_\alpha(t)} \gamma(t) F(t,\alpha)|| dt \leq M \frac{1}{Re(\lambda)-\omega}. \end{aligned}$$

□

Definition 3.7. An $\Re_\lambda^c$ application that verifies

$$\Re_\lambda^c - \Re_\mu^c = (\mu-\lambda)\Re_\lambda^c \Re_\mu^c$$

is called a conformable pseudo-resolvent.

Example 3.8. Consider the $\chi_F^\alpha-$semi-group defined in Example 3.5. We have

$$\Re_\lambda^c(\gamma(t)) = \frac{1}{\lambda-\alpha+1}$$

and

$$\Re_\lambda^c - \Re_\mu^c = \frac{1}{\lambda-\alpha+1} - \frac{1}{\mu-\alpha+1} = \frac{\mu-\lambda}{(\lambda-\alpha+1)(\mu-\alpha+1)} = (\mu-\lambda)\Re_\lambda^c \Re_\mu^c.$$

Then it's a conformable pseudo-resolvent.

Remark 3.9. Let $\alpha \in (0,1]$.
1. $\frac{1}{\mu} = \int_0^{+\infty} e^{-\mu G_\alpha(t)} F(t,\alpha) dt$.
2. $\lim_{h\to 0} G_\alpha(h) = F(0,\alpha)$.

Theorem 3.10. *For all $\lambda \in \{\lambda \in \mathbb{C} : Re(\lambda) > \omega\}$. The operator $\Re_\lambda^c$ is a conformable pseudo-resolvent.*

Proof. By Remark 3.9

$$\begin{aligned} \frac{1}{\mu-\lambda}\Re_\lambda^c - \frac{1}{\mu-\lambda}\Re_\mu^c &= \int_0^{+\infty} e^{-(\mu-\lambda)G_\alpha(t)} \Re_\lambda^c F(t,\alpha) dt - \int_0^{+\infty} e^{-(\mu-\lambda)G_\alpha(t)} \Re_\mu^c F(t,\alpha) dt \\ &= I_1 - I_2. \end{aligned}$$

Where

$$I_1 = \int_0^{+\infty} \int_0^{+\infty} e^{-(\mu-\lambda)G_\alpha(t)} e^{-\lambda G_\alpha(s)} \gamma(s) F(t,\alpha) F(s,\alpha) dt ds$$

and

$$I_2 = \int_0^{+\infty} \int_0^{+\infty} e^{-(\mu-\lambda)G_\alpha(t)} e^{-\mu G_\alpha(s)} \gamma(s) F(t,\alpha) F(s,\alpha) dt ds.$$

We have

$$\begin{aligned} I_2 &= \int_0^{+\infty}\int_0^{+\infty} e^{-(\mu-\lambda)G_\alpha(t)}e^{-\mu G_\alpha(s)}\gamma(s)F(t,\alpha)F(s,\alpha)dtds \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-(\mu-\lambda)(G_\alpha(t)+G_\alpha(s))}e^{-\lambda G_\alpha(s)}\gamma(s)F(t,\alpha)F(s,\alpha)dtds \\ &= \int_0^{+\infty}\int_s^{+\infty} e^{-(\mu-\lambda)G_\alpha(u)}e^{-\lambda G_\alpha(s)}\gamma(s)F(u,\alpha)F(s,\alpha)duds. \end{aligned}$$

Then

$$\begin{aligned} I_1 - I_2 &= \int_0^{+\infty}\int_0^{s} e^{-(\mu-\lambda)G_\alpha(u)}e^{-\lambda G_\alpha(s)}\gamma(s)F(u,\alpha)F(s,\alpha)duds \\ &= \int_0^{+\infty}\int_u^{+\infty} e^{-(\mu-\lambda)G_\alpha(u)}e^{-\lambda G_\alpha(s)}\gamma(s)F(u,\alpha)F(s,\alpha)dsdu \\ &= \int_0^{+\infty} e^{-(\mu-\lambda)G_\alpha(u)}F(u,\alpha)\left(\int_u^{+\infty} e^{-\lambda G_\alpha(s)}\gamma(s)F(s,\alpha)ds\right)du \\ &= \int_0^{+\infty} e^{-\mu G_\alpha(u)}F(u,\alpha)\left(\int_u^{+\infty} e^{-\lambda[G_\alpha(s)-G_\alpha(u)]}\gamma(s)F(s,\alpha)ds\right)du \\ &= \int_0^{+\infty} e^{-\mu G_\alpha(u)}F(u,\alpha)\left(\int_0^{+\infty} e^{-\lambda G_\alpha(r)}\gamma\left(G_\alpha^{-1}[G_\alpha(r)+G_\alpha(u)]\right)F(r,\alpha)dr\right)du \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-\mu G_\alpha(u)}e^{-\lambda G_\alpha(r)}\gamma\left(G_\alpha^{-1}[G_\alpha(r)+G_\alpha(u)]\right)F(r,\alpha)F(u,\alpha)drdu. \end{aligned}$$

$\{\gamma(t)\}_{t\geq 0}$ is the conformable semi-group, then by Definition 2.7

$$\begin{aligned} &\frac{1}{\mu-\lambda}\Re_\lambda^c - \frac{1}{\mu-\lambda}\Re_\mu^c = I_1 - I_2 \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-\mu G_\alpha(u)}e^{-\lambda G_\alpha(r)}\gamma\left(G_\alpha^{-1}[G_\alpha(r)+G_\alpha(u)]\right)F(r,\alpha)F(u,\alpha)drdu \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-\mu G_\alpha(u)}e^{-\lambda G_\alpha(r)}\gamma(r+u)F(r,\alpha)F(u,\alpha)drdu \\ &= \Re_\lambda^c\Re_\mu^c. \end{aligned}$$

□

Definition 3.11. The conformable resolving set of the operator Γ is defined by

$$\Re^c(\nu,\Gamma) = (\nu I - \Gamma)^{-1}$$

where $\nu \in \{\nu \in \mathbb{C} : (\nu I - \Gamma) \text{ is invertible}\}$.

Theorem 3.12. *Let $(\gamma(t))_{t\geq 0}$ be a conformable semi-group, with generator Γ. Then:*

$$\Re_\nu^c = \Re^c(\nu,\Gamma).$$

Proof. If $x \in D(\Gamma)$. By (2.1) and Theorem 2.2

$$\begin{aligned} \Re_\nu^c Ax &= \int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma(t)\Gamma x F(t,\alpha)dt = \int_0^{+\infty} e^{-\nu G_\alpha(t)}\chi_F^\alpha(\gamma(t)x)F(t,\alpha)dt \\ &= \int_0^{+\infty} e^{-\nu G_\alpha(t)}(\gamma(t)x)'dt = \nu\int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma(t)xdt - x = \nu\Re_\nu^c x - x. \end{aligned}$$

Then

$$(\nu I - \Gamma)\Re_\nu^c x = x$$

and

$$\Re_\nu^c = \Re^c(\nu, \Gamma).$$

Let $x \in X$ and λ such that $Re(\nu) > \omega$.

$$\begin{aligned}
&\frac{1}{h}\left(\gamma(\frac{h}{F(0,\alpha)})\Re_\nu^c x - \Re_\nu^c x\right) = \frac{1}{h}\int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma(\frac{h}{F(0,\alpha)})\gamma(t)xF(t,\alpha)dt \\
&- \frac{1}{h}\int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma(t)xF(t,\alpha)dt \\
&= \frac{1}{F(0,\alpha)h}\left(\int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma(h)\gamma(t)xF(t,\alpha)dt - \int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma(t)xF(t,\alpha)dt\right).
\end{aligned}$$

By Definition 2.7

$$\begin{aligned}
\int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma(h)\gamma(t)xF(t,\alpha)dt &= \int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma\left(G_\alpha^{-1}[G_\alpha(h) + G_\alpha(t)]\right)xF(t,\alpha)dt \\
&= \int_h^{+\infty} e^{-\nu[G_\alpha(u) - G_\alpha(h)]}\gamma(u)xF(u,\alpha)du.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{h}\left(\gamma(\frac{h}{F(0,\alpha)})\Re_\nu^c x - \Re_\nu^c x\right) = \\
&= \frac{1}{F(0,\alpha)}\left(\frac{e^{\nu G_\alpha(h)} - 1}{h}\Re_\nu^c x - \frac{e^{\nu G_\alpha(h)}}{h}\int_0^h e^{-\nu G_\alpha(t)}\gamma(t)xF(t,\alpha)dt\right).
\end{aligned}$$

By Remark 3.9 and (2.1)

$$\lim_{h\to 0}\frac{1}{h}\left(\gamma(\frac{h}{F(0,\alpha)})\Re_\nu^c x - \Re_\nu^c x\right) = \frac{1}{F(0,\alpha)}[F(0,\alpha)\nu\Re_\nu^c x - F(0,\alpha)x] = \nu\Re_\nu^c x - x$$

Then

$$\Gamma\Re_\nu^c x = \nu\Re_\nu^c x - x$$

Thus

$$(\nu I - \Gamma)\Re_\nu^c x = x.$$

□

4. Conformable Integrated Semi-groups

To better understand some properties, please refer to the properties known as Fubini/Tonelli or dominated convergence, which cannot be mentioned here.

Lemma 4.1. *Let $\{\gamma(t)\}_{t\geq 0}$ be a conformable $\chi_F^\alpha-$exponentially bounded semi-group and Γ its infinitesimal generator. Let $\mathfrak{S}(t) = \int_0^t \gamma(s)F(s,\alpha)ds$. Then*
$\Re_\nu^c(\mathfrak{S}(t)) = \frac{1}{\nu}\Re_\nu^c(\gamma(t))$.

Proof. By Definition 3.4

$$\begin{aligned}
\Re_\nu^c(S(t)) &= \int_0^{+\infty} e^{-\nu G_\alpha(t)}\mathfrak{S}(t)F(t,\alpha)dt = \int_0^{+\infty} e^{-\nu G_\alpha(t)}\int_0^t \gamma(s)F(s,\alpha)dsF(t,\alpha)dt \\
&= \frac{1}{\nu}\int_0^{+\infty} e^{-\nu G_\alpha(t)}\gamma(t)F(t,\alpha)dt = \frac{1}{\nu}\Re_\nu^c[\gamma(t)].
\end{aligned}$$

□

Lemma 4.2. *Let $\{\gamma(t)\}_{t\geq 0}$ be a conformable $\chi_F^{\alpha}-$exponentially bounded semi-group and*

$$\Re_{\nu}^{c}=\nu\int_0^{+\infty}e^{-\nu G_{\alpha}(t)}\mathfrak{S}(t)F(t,\alpha)dt$$

Then
$\frac{1}{\mu}\frac{1}{\mu-\lambda}[\frac{\Re_{\lambda}^{c}}{\lambda}-\frac{\Re_{\mu}^{c}}{\mu}]$
$=\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda G_{\alpha}(v)}e^{-\mu G_{\alpha}(u)}F(v,\alpha)F(u,\alpha)\left(\int_v^{G_{\alpha}^{-1}[G_{\alpha}(v)+G_{\alpha}(u)]}\mathfrak{S}(r)F(r,\alpha)dr\right)dudv.$

Proof.

$$\begin{aligned}
&\frac{1}{\mu-\lambda}[\frac{\Re_{\lambda}^{c}}{\lambda}-\frac{\Re_{\mu}^{c}}{\mu}]=\int_0^{+\infty}e^{-(\mu-\lambda)G_{\alpha}(t)}F(t,\alpha)[\frac{\Re_{\lambda}^{c}}{\lambda}-\frac{\Re_{\mu}^{c}}{\mu}]dt\\
&=\int_0^{+\infty}e^{-(\mu-\lambda)G_{\alpha}(t)}F(t,\alpha)[\int_0^{+\infty}e^{-\lambda G_{\alpha}(r)}\mathfrak{S}(r)F(r,\alpha)dr\\
&-\int_0^{+\infty}e^{-\mu G_{\alpha}(r)}\mathfrak{S}(r)F(r,\alpha)dr]dt\\
&=\int_0^{+\infty}\int_0^{+\infty}e^{-(\mu-\lambda)G_{\alpha}(t)}e^{-\lambda G_{\alpha}(r)}\mathfrak{S}(r)F(t,\alpha)F(r,\alpha)drdt\\
&-\int_0^{+\infty}\int_0^{+\infty}e^{-(\mu-\lambda)G_{\alpha}(t)}e^{-\mu G_{\alpha}(r)}\mathfrak{S}(r)F(t,\alpha)F(r,\alpha)drdt\\
&=\int_0^{+\infty}\int_0^{+\infty}e^{-(\mu-\lambda)G_{\alpha}(t)}e^{-\lambda G_{\alpha}(r)}\mathfrak{S}(r)F(t,\alpha)F(r,\alpha)dtdr\\
&-\int_0^{+\infty}\int_0^{+\infty}e^{-(\mu-\lambda)[G_{\alpha}(t)+G_{\alpha}(r)]}e^{-\lambda G_{\alpha}(r)}\mathfrak{S}(r)F(t,\alpha)F(r,\alpha)dtdr\\
&=\int_0^{+\infty}\int_0^{+\infty}e^{-(\mu-\lambda)G_{\alpha}(t)}e^{-\lambda G_{\alpha}(r)}\mathfrak{S}(r)F(t,\alpha)F(r,\alpha)dtdr\\
&-\int_0^{+\infty}\int_r^{+\infty}e^{-(\mu-\lambda)G_{\alpha}(u)}e^{-\lambda G_{\alpha}(r)}\mathfrak{S}(r)F(u,\alpha)F(r,\alpha)dudr\\
&=\int_0^{+\infty}\int_0^{r}e^{-(\mu-\lambda)G_{\alpha}(u)}e^{-\lambda G_{\alpha}(r)}\mathfrak{S}(r)F(u,\alpha)F(r,\alpha)dudr\\
&=\int_0^{+\infty}\int_u^{+\infty}e^{-(\mu-\lambda)G_{\alpha}(u)}e^{-\lambda G_{\alpha}(r)}\mathfrak{S}(r)F(u,\alpha)F(r,\alpha)drdu\\
&=\int_0^{+\infty}e^{-\mu G_{\alpha}(u)}F(u,\alpha)\int_u^{+\infty}e^{-\lambda[G_{\alpha}(r)-G_{\alpha}(u)]}\mathfrak{S}(r)F(r,\alpha)drdu\\
&=\int_0^{+\infty}e^{-\mu G_{\alpha}(u)}F(u,\alpha)\int_u^{+\infty}e^{-\lambda G_{\alpha}(v)}\mathfrak{S}\left(G_{\alpha}^{-1}[G_{\alpha}(v)+G_{\alpha}(u)]\right)F(v,\alpha)dvdu\\
&=\int_0^{+\infty}\int_0^{+\infty}e^{-\mu G_{\alpha}(u)}e^{-\lambda G_{\alpha}(v)}\mathfrak{S}\left(G_{\alpha}^{-1}[G_{\alpha}(v)+G_{\alpha}(u)]\right)F(v,\alpha)F(u,\alpha)dvdu.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{\mu-\lambda}\left[\frac{\Re_{\lambda}^{c}}{\lambda}-\frac{\Re_{\mu}^{c}}{\mu}\right]\\
&=\int_0^{+\infty}\int_0^{+\infty}e^{-\mu G_{\alpha}(u)}e^{-\lambda G_{\alpha}(v)}\mathfrak{S}\left(G_{\alpha}^{-1}[G_{\alpha}(v)+G_{\alpha}(u)]\right)F(v,\alpha)F(u,\alpha)dvdu. \quad (4.2)
\end{aligned}$$

By Remark 3.9 and equation (4.2)

$$\frac{1}{\mu}\frac{1}{\mu-\lambda}\Big[\frac{\Re_\lambda^c}{\lambda}-\frac{\Re_\mu^c}{\mu}\Big]=\int_0^{+\infty}e^{-\mu G_\alpha(t)}F(t,\alpha)\frac{1}{\mu-\lambda}\Big[\frac{\Re_\lambda^c}{\lambda}-\frac{\Re_\mu^c}{\mu}\Big]dt$$
$$=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}e^{-\mu[G_\alpha(u)+G_\alpha(t)]}e^{-\lambda G_\alpha(v)}\mathfrak{S}\left(G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]\right)$$
$$\times F(v,\alpha)F(u,\alpha)F(t,\alpha)dvdudt$$
$$=\int_0^{+\infty}e^{-\lambda G_\alpha(v)}\mathfrak{S}\left(G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]\right)F(v,\alpha)$$
$$\times\int_0^{+\infty}\int_0^{+\infty}e^{-\mu[G_\alpha(u)+G_\alpha(t)]}F(t,\alpha)F(u,\alpha)dtdudv$$
$$=\int_0^{+\infty}e^{-\lambda G_\alpha(v)}\mathfrak{S}\left(G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]\right)F(v,\alpha)$$
$$\times\int_0^{+\infty}\int_u^{+\infty}e^{-\mu G_\alpha(t)}F(t,\alpha)F(u,\alpha)dtdudv$$
$$=\int_0^{+\infty}e^{-\lambda G_\alpha(v)}F(v,\alpha)$$
$$\times\int_0^{+\infty}\int_u^{+\infty}e^{-\mu G_\alpha(t)}\mathfrak{S}\left(G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]\right)F(t,\alpha)F(u,\alpha)dtdudv$$
$$=\int_0^{+\infty}e^{-\lambda G_\alpha(v)}F(v,\alpha)$$
$$\times\int_0^{+\infty}\int_0^{t}e^{-\mu G_\alpha(t)}\mathfrak{S}\left(G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]\right)F(t,\alpha)F(u,\alpha)dudtdv$$
$$=\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda G_\alpha(v)}e^{-\mu G_\alpha(t)}F(v,\alpha)F(t,\alpha)$$
$$\times\left(\int_0^t\mathfrak{S}\left(G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]\right)F(u,\alpha)du\right)dtdv$$
$$=\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda G_\alpha(v)}e^{-\mu G_\alpha(t)}F(v,\alpha)F(t,\alpha)$$
$$\times\left(\int_v^{G_\alpha^{-1}[G_\alpha(v)+G_\alpha(t)]}\mathfrak{S}(r)F(r,\alpha)dr\right)dtdv.$$

Then

$$\frac{1}{\mu}\frac{1}{\mu-\lambda}\Big[\frac{\Re_\lambda^c}{\lambda}-\frac{\Re_\mu^c}{\mu}\Big]=\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda G_\alpha(v)}e^{-\mu G_\alpha(u)}F(v,\alpha)F(u,\alpha)$$
$$\times\left(\int_v^{G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]}\mathfrak{S}(r)F(r,\alpha)dr\right)dudv.$$

□

Lemma 4.3. *Let $\{\gamma(t)\}_{t\geq 0}$ be a conformable $\chi_F^\alpha-$exponentially bounded semi-group and*

$$\Re_\nu^c=\nu\int_0^{+\infty}e^{-\nu G_\alpha(t)}\mathfrak{S}(t)F(t,\alpha)dt$$

Then

$$\frac{1}{\lambda\mu}\frac{\Re_\mu^c}{\mu}=\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda G_\alpha(v)}e^{-\mu G_\alpha(u)}F(v,\alpha)F(u,\alpha)\left(\int_0^u\mathfrak{S}(r)F(r,\alpha)dr\right)dudv.$$

Proof. By remark 3.9

$$\begin{aligned}\frac{1}{\lambda\mu}\frac{\Re_\mu^c}{\mu}&=\int_0^{+\infty}e^{-\lambda G_\alpha(v)}F(v,\alpha)\int_0^{+\infty}e^{-\mu G_\alpha(r)}F(r,\alpha)\int_0^{+\infty}e^{-\mu G_\alpha(t)}\mathfrak{S}(t)F(t,\alpha)dtdrdv\\&=\int_0^{+\infty}e^{-\lambda G_\alpha(v)}F(v,\alpha)\int_0^{+\infty}\mathfrak{S}(t)F(t,\alpha)\int_0^{+\infty}e^{-\mu[G_\alpha(t)+G_\alpha(r)]}F(r,\alpha)drdtdv\\&=\int_0^{+\infty}e^{-\lambda G_\alpha(v)}F(v,\alpha)\int_0^{+\infty}\mathfrak{S}(t)F(t,\alpha)\int_t^{+\infty}e^{-\mu G_\alpha(u)}F(u,\alpha)dudtdv\\&=\int_0^{+\infty}e^{-\lambda G_\alpha(v)}F(v,\alpha)\int_0^{+\infty}e^{-\mu G_\alpha(u)}\int_0^u\mathfrak{S}(t)F(t,\alpha)F(u,\alpha)dtdudv.\end{aligned}$$

Then

$$\frac{1}{\lambda\mu}\frac{\Re_\mu^c}{\mu}=\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda G_\alpha(v)}e^{-\mu G_\alpha(u)}F(v,\alpha)F(u,\alpha)\left(\int_0^u\mathfrak{S}(r)F(r,\alpha)dr\right)dudv.$$

□

Theorem 4.4. *Let $\mathfrak{S}:[0,\infty)\to B(X)$ be a strongly continuous map for which there exists $M\geq 0$ and $\omega\in\mathbb{R}$ such that*

$$||\mathfrak{S}(t)||\leq Me^{\omega G_\alpha(t)},\forall t\geq 0$$

The following statements are equivalent:
1. $\Re_\lambda^c=\lambda\int_0^{+\infty}e^{-\lambda G_\alpha(t)}\mathfrak{S}(t)F(t,\alpha)dt$ is a conformable pseudo-resolvant.
2. For all $t,s\geq 0$, we have

$$\mathfrak{S}(t)\mathfrak{S}(s)=\int_t^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]}\mathfrak{S}(r)F(r,\alpha)dr-\int_0^s\mathfrak{S}(r)F(r,\alpha)dr.$$

Proof. We have

$$\frac{1}{\mu\lambda}\frac{1}{\mu-\lambda}[\Re_\lambda^c-\Re_\mu^c]=\frac{1}{\mu}\frac{1}{\mu-\lambda}[\frac{\Re_\lambda^c}{\lambda}-\frac{\Re_\mu^c}{\mu}]-\frac{1}{\lambda\mu}\frac{\Re_\mu^c}{\mu}$$

and by Lemmas 4.2 and 4.3

$$\begin{aligned}&\frac{1}{\mu\lambda}\frac{1}{\mu-\lambda}[\Re_\lambda^c-\Re_\mu^c]=\\&\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda G_\alpha(v)}e^{-\mu G_\alpha(u)}F(v,\alpha)F(u,\alpha)\\&\times\left(\int_v^{G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]}\mathfrak{S}(r)F(r,\alpha)dr-\int_0^u\mathfrak{S}(r)F(r,\alpha)dr\right)dudv.\end{aligned}$$

On the other hand

$$\frac{\Re_\lambda^c}{\lambda}\frac{\Re_\mu^c}{\mu}=\int_0^{+\infty}\int_0^{+\infty}e^{-\lambda G_\alpha(v)}e^{-\mu G_\alpha(u)}\mathfrak{S}(u)\mathfrak{S}(v)F(v,\alpha)F(u,\alpha)dudv.$$

Then
$\Re_\lambda^c$ is a conformable pseudo-resolvant if and only if

$$\frac{1}{\mu\lambda}\frac{1}{\mu-\lambda}[\Re_\lambda^c-\Re_\mu^c]=\frac{\Re_\lambda^c}{\lambda}\frac{\Re_\mu^c}{\mu}$$

Thus $\Re_\lambda^c$ is a conformable pseudo-resolvant if and only if

$$\mathfrak{S}(v)\mathfrak{S}(u) = \int_v^{G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]} \mathfrak{S}(r)F(r,\alpha)dr - \int_0^u \mathfrak{S}(r)F(r,\alpha)dr.$$

□

Lemma 4.5. *We have*
$\mathfrak{S}(t)\mathfrak{S}(s) = \mathfrak{S}(s)\mathfrak{S}(t)$,

$$\mathfrak{S}(t)\mathfrak{S}(s) = \int_0^t \left(\mathfrak{S}(G_\alpha^{-1}[G_\alpha(\tau)+G_\alpha(s)]) - \mathfrak{S}(\tau)\right) F(\tau,\alpha)d\tau$$

and

$$\mathfrak{S}(t)\mathfrak{S}(0) = 0, \forall t \geq 0.$$

Proof.

$$\begin{aligned}
\mathfrak{S}(t)\mathfrak{S}(s) &= \int_t^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]} \mathfrak{S}(r)F(r,\alpha)dr - \int_0^s \mathfrak{S}(r)F(r,\alpha)dr \\
&= \int_0^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]} \mathfrak{S}(r)F(r,\alpha)dr - \int_0^t \mathfrak{S}(r)F(r,\alpha)dr - \int_0^s \mathfrak{S}(r)F(r,\alpha)dr \\
&= \int_s^{G_\alpha^{-1}[G_\alpha(s)+G_\alpha(t)]} \mathfrak{S}(r)F(r,\alpha)dr - \int_0^t \mathfrak{S}(r)F(r,\alpha)dr \\
&= \mathfrak{S}(s)\mathfrak{S}(t).
\end{aligned}$$

and put $G_\alpha(\tau) = G_\alpha(r) - G_\alpha(s)$

$$\begin{aligned}
\mathfrak{S}(t)\mathfrak{S}(s) &= \int_s^{G_\alpha^{-1}[G_\alpha(s)+G_\alpha(t)]} \mathfrak{S}(r)F(r,\alpha)dr - \int_0^t \mathfrak{S}(r)F(r,\alpha)dr \\
&= \int_0^t \mathfrak{S}(G_\alpha^{-1}[G_\alpha(s)+G_\alpha(\tau))F(\tau,\alpha)d\tau - \int_0^t \mathfrak{S}(r)F(r,\alpha)dr \\
&= \int_0^t \left(\mathfrak{S}(G_\alpha^{-1}[G_\alpha(\tau)+G_\alpha(s)]) - \mathfrak{S}(\tau)\right) F(\tau,\alpha)d\tau.
\end{aligned}$$

□

Example 4.6. Consider the $\nu_F^\alpha-$semi-group defined in Example 3.5. We have

$$\mathfrak{S}(t) = \frac{1}{\alpha-1}\left(e^{(\alpha-1)G_\alpha(t)} - 1\right)$$

Then $\mathfrak{S}(t)\mathfrak{S}(s) = \mathfrak{S}(s)\mathfrak{S}(t)$, $\mathfrak{S}(t)\mathfrak{S}(0) = 0$ and

$$\mathfrak{S}(t)\mathfrak{S}(s) = \frac{1}{(\alpha-1)^2}\left(e^{(\alpha-1)G_\alpha(t)} - 1\right)\left(e^{(\alpha-1)G_\alpha(s)} - 1\right)$$

On the other hand

$$\begin{aligned}
&\int_t^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]} \mathfrak{S}(r)F(r,\alpha)dr = \frac{1}{\alpha-1}\int_t^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]} \left(e^{(\alpha-1)G_\alpha(r)} - 1\right) F(r,\alpha)dr \\
&= \frac{1}{\alpha-1}\left(\int_t^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]} e^{(\alpha-1)G_\alpha(r)}F(r,\alpha)dr - \int_t^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]} F(r,\alpha)dr\right) \\
&= \frac{1}{\alpha-1}\left(\int_{G_\alpha(t)}^{G_\alpha(t)+G_\alpha(s)} e^{(\alpha-1)u}du - \frac{\alpha}{2}\int_t^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]} (e^r + e^{-r})dr\right) \\
&= \frac{1}{(\alpha-1)^2}\left(e^{(\alpha-1)G_\alpha(s)} - 1\right)e^{(\alpha-1)G_\alpha(t)} - \frac{G_\alpha(s)}{\alpha-1}.
\end{aligned}$$

and

$$\int_0^s \mathfrak{S}(r)F(r,\alpha)dr = \frac{1}{\alpha-1}\int_0^s \left(e^{(\alpha-1)G_\alpha(r)}-1\right)F(r,\alpha)dr$$
$$= \frac{1}{\alpha-1}\left(\int_0^s e^{(\alpha-1)G_\alpha(r)}F(r,\alpha)dr - \int_0^s F(r,\alpha)dr\right)$$
$$= \frac{1}{\alpha-1}\left(\int_0^{G_\alpha(s)} e^{(\alpha-1)u}du - \frac{\alpha}{2}\int_0^s (e^r+e^{-r})dr\right)$$
$$= \frac{1}{(\alpha-1)^2}\left(e^{(\alpha-1)G_\alpha(s)}-1\right) - \frac{G_\alpha(s)}{\alpha-1}.$$

Finaly

$$\int_t^{G_\alpha^{-1}[G_\alpha(t)+G_\alpha(s)]} \mathfrak{S}(r)F(r,\alpha)dr - \int_0^s \mathfrak{S}(r)F(r,\alpha)dr$$
$$= \frac{1}{(\alpha-1)^2}\left(e^{(\alpha-1)G_\alpha(t)}-1\right)\left(e^{(\alpha-1)G_\alpha(s)}-1\right)$$
$$= \mathfrak{S}(t)\mathfrak{S}(s).$$

5. Conclusion and Future Work

A new definition of conformable pseudo-resolvant has been given. A relationship between the conformable Laplace transform of semi-group and the resolvent set have been established. From Theorem 4.4, we define the conformable integrated semigroup $\mathfrak{S}$ with the following:
i) $\mathfrak{S}(0) = 0$,
ii) The application $t \to \mathfrak{S}(t)$ is strongly continuous,
iii) $\mathfrak{S}(v)\mathfrak{S}(u) = \int_v^{G_\alpha^{-1}[G_\alpha(v)+G_\alpha(u)]} \mathfrak{S}(r)F(r,\alpha)dr - \int_0^u \mathfrak{S}(r)F(r,\alpha)dr$.
We will discuss the characteristics of this concept as the resolvent set and the resolvent and other properties.

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