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CHARACTERIZATION OF TAUBERIAN AND CO-TAUBERIAN PERTURBATION OF 2×2 BLOCK OPERATOR MATRICES

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Abstract. In this paper, we introduce new classes of bounded linear operators, called $\mathcal{T}$-singular, $\mathcal{T}^d$-cosingular and $\mathcal{T}$-Riesz operators. Using these classes, some characterizations of tauberian and co-tauberian perturbation classes are established. Also, several results on tauberian and co-tauberian theory of a bounded matrix operator are presented.

1. Introduction

For Banach spaces X and Y, we use $\mathcal{L}(X,Y)$ to represent the set of all bounded linear operators from X to Y. For $T \in \mathcal{L}(X,Y)$, we denote by $\mathcal{R}(T), Y/\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range, the co-kernel and the kernel of T respectively. The dual space (resp., the bidual) is symbolized as X^* (resp., X^{**}), where T^* stands for the adjoint of an operator T, and T^{**} is the second adjoint.

Tauberian operators have been useful in the study of real interpolation theory of (non reflexive) Banach spaces as well as some questions related to the preservation of isomorphic properties between those spaces [17]. They also have some similar actions on weakly compact sets as upper semi Fredholm operators do on compact sets. In 1991, T. Álvarez and M. Gonzàlez in give the word tauberian to describe those bounded operators $T : X \to Y$ that are realized $T^{**-1}(Y) \subseteq X$. For the study of the duality problem associated with tauberian operators, M. González and V. M. Onieva introduced the concept of co-tauberian operators in [12]. Specifically, a bounded linear operator T from X to Y is termed co-tauberian, $T \in \mathcal{T}^d(X,Y)$, if T^* is tauberian, see [10] for more details.

The concept of weakly demicompact operator was introduced by B. Krichen and D. O'Regan in [16]. A linear operator $T : \mathcal{D}(T) \subset X \longrightarrow Y$ is said to be weakly demicompact, $T \in \mathcal{WDC}(X,Y)$, if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n - Tx_n$ weakly converges in X, there exists a weakly convergent subsequence of $(x_n)_n$. Obviously, weakly compact operators are weakly demicompact. For $T \in \mathcal{L}(X,Y)$, it is proved in [10] that $T \in \mathcal{WDC}(X,Y)$ if and only if, $I - T \in \mathcal{T}(X,Y)$. We refer to [7, 16] for further information on these class of operators.

Recall also an operator $T \in \mathcal{L}(X,Y)$ is called upper (resp. lower) semi-Fredholm if $\mathcal{R}(T)$ is closed in Y and $\dim(\mathcal{N}(T)) < \infty$ (resp. $\mathcal{R}(T)$ is closed in Y and $\dim(Y/\mathcal{R}(T)) < \infty$). Therefore, T is said to be Fredholm if it is both upper semi-Fredholm and lower semi-Fredholm operator. According to [10], if a Banach space X has no reflexive infinite-dimensional subspaces (resp. if Y has no infinite-dimensional reflexive quotients subspaces), the upper (resp. lower) semi-Fredholm operators from X to Y, serve as trivial examples of tauberian (resp. co-tauberian) operators. It results, under these conditions, that the set of upper (resp. lower) semi-Fredholm perturbation coincides with the set of tauberian (resp. co-tauberian) perturbation.

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In [19], K. W. Yang introduced the class of generalized Fredholm operators for bounded linear operators acting on Banach space by $\Phi_g(X,Y) := \Phi_{g+}(X,Y) \cap \Phi_{g-}(X,Y)$, where

$$\Phi_{g+}(X,Y) := \{T \in \mathcal{L}(X,Y) \,:\, \mathcal{N}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\}, \text{ and}$$

$$\Phi_{g-}(X,Y) := \{T \in \mathcal{L}(X,Y) \,:\, Y/\mathcal{R}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\}$$

denote respectively the sets of upper and lower generalized semi-Fredholm operators which are defined in [4]. It is well known that the class of upper (resp. lower) generalized semi-Fredholm operators acting on Banach spaces is strictly contained in the class of tauberian (resp. co-tauberian) operators (see [4, 14]). For more information on generalized semi-Fredholm operators we refer the reader to [3, 8].

We say that $T \in \mathcal{L}(X,Y)$ is weakly precompact, if (Tx_n) has a weakly Cauchy subsequence for every bounded sequence (x_n) in X. For ν is a purely atomic measure, every tauberian from $L_1(\nu)$ to Y is upper semi-Fredholm [12]. However, when μ is a non-purely atomic finite measure, the set of weakly precompact operators from $L_1(\mu)$ to Y coincides with the perturbation class of $\mathcal{T}(L_1(\mu),Y)$, see [11]. Authors in [11] proved that $T \in \mathcal{L}(L_1(\mu),Y)$ is tauberian if and only if, $\beta_T > 0$, where

$$\beta_T := \inf\{\liminf_n \|Tf_n\| : (f_n)_n \subset L_1(\mu) \text{ normalized and disjoint}\}.$$

For more details on this quantity we refer the reader to [10].

Now, recall that for all $\lambda \in \mathbb{C}\backslash\{0\}$, a bounded linear operator T acting from X into Y is said to be Riesz (resp. g-Riesz) operator, $T \in \mathcal{R}(X,Y)$ (resp. $T \in \mathcal{R}_g(X,Y)$), if $(\lambda - T) \in \Phi(X,Y)$ (resp. $\in \Phi_g(X,Y)$) (see [18] and [4, 19]).

Perturbation theory, as rigorously developed by [15], is a cornerstone of linear analysis, concerned with understanding how the properties of a linear operator change when it is subjected to a small perturbation. The classical theory, initiated by Kato and others, investigates stability of spectral and analytical properties under additive or multiplicative perturbations. The study of classical perturbation classes, such as strictly singular and strictly cosingular operators, has been extensively developed, particularly for semi-Fredholm and Fredholm operators [1, 15, 18]. More recently, the focus has shifted to broader operator classes. Tauberian and co-tauberian operators, introduced as tools in real interpolation theory [17] and characterized by conditions on their second adjoint and adjoint, respectively, have been shown to share many stability properties with semi-Fredholm operators but apply to a wider range of Banach spaces [10, 12]. Specifically, an operator T is tauberian ($T \in \mathcal{T}$) if $T^{**-1}(Y) \subseteq X$, and co-tauberian ($T \in \mathcal{T}^d$) if its adjoint T^* is tauberian [10, 12]. Understanding the perturbation classes associated with these operators, denoted $\mathcal{PT}$ and $\mathcal{PT}^d$, is a natural next step.

The analysis becomes particularly rich when considering operators acting on product spaces, where they admit a 2×2 block matrix representation. Indeed, any bounded linear operator on a product Banach space $X_1 \times X_2$ can be written in matrix form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with entries in appropriate spaces. Perturbation and spectral problems for such operator matrices are central to numerous applications, from differential equations to systems theory. While perturbation results for semi-Fredholm block matrices have been studied [9], the analogous theory for tauberian and co-tauberian operators remains less developed. This paper aims to fill this gap by introducing new operator classes called $\mathcal{T}$-singular, $\mathcal{T}^d$-cosingular, and $\mathcal{T}$-Riesz operators to characterize the perturbation classes $\mathcal{PT}$ and $\mathcal{PT}^d$. We extend classical results by showing, for instance, that under appropriate isomorphisms, the set of tauberian perturbations coincides with $\mathcal{T}$-singular operators, and we establish stability properties for these perturbation classes on product spaces. A key

contribution is the analysis of upper-triangular operator matrices $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where we provide conditions ensuring that M_C inherits tauberian or co-tauberian properties from its diagonal entries A and B (Lemma 4.1) and conversely.

Finally, the spectral theory of operator matrices is deeply connected to these perturbation concepts. We define the tauberian and co-tauberian spectra of an operator A as $\sigma_{\mathcal{T}}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \notin \mathcal{T}\}$ and similarly for $\mathcal{T}^d$. A central result of this paper (Theorem 4.1) demonstrates that for the block matrix M_C, the tauberian spectrum is contained in the union of the tauberian spectra of its diagonal entries: $\sigma_{\mathcal{T}}(M_C) \subset \sigma_{\mathcal{T}}(A) \cup \sigma_{\mathcal{T}}(B)$. An analogous inclusion holds for the co-tauberian spectrum. This inclusion provides a powerful tool for localizing the "tauberian non-regularity" of the whole matrix in terms of its diagonal components. Moreover, we introduce the class of $\mathcal{T}$-Riesz operators $\mathcal{R}_{\mathcal{T}}$ (Definition 4.1) and show that A and B are $\mathcal{T}$-Riesz if and only if $\lambda - M_C$ is tauberian for all nonzero λ not in the spectrum of B (Lemma 4.3). These results extend the well-known spectral inclusions for Fredholm and generalized Fredholm operators to the tauberian framework, offering new insights into the structure of block operator matrices on Banach spaces. For further background on tauberian and co-tauberian operators, their spectra, and related perturbation classes, we refer the reader to the comprehensive monograph by Gonzàlez and Martínez-Abejón [10], as well as foundational works by Gonzàlez and Onieva [12] and more recent developments by Azzouz, Beghdadi, and Krichen [6, 7].

Now, let us outline the content of this paper. In Section 2, we introduce and study new classes of operators and we give several results on tauberian and co-tauberian perturbations classes. In Section 3, we extend some results in [1, 13] to tauberian and co-tauberian perturbations theory on product of Banach spaces. Moreover, for μ is a non-purely atomic, finite measure, we show that the set $\mathcal{TS}(L_1(\mu), Y)$ is a closed in $\mathcal{L}(L_1(\mu), Y)$. In Section 4, we establish some characterizations of tauberian and co-tauberian 2×2 operator matrices on Banach spaces.

2. Some results on tauberian and co-tauberian perturbation classes

In this section, we investigate some results concerning the class of tauberian and co-tauberian perturbations. We start by presenting the following definition.

Definition 2.1. Let X and Y be two Banach spaces and let $K \in \mathcal{L}(X, Y)$. K is called

(i) Tauberian perturbation if $U + K \in \mathcal{T}(X, Y)$, whenever $U \in \mathcal{T}(X, Y)$.

(ii) Co-tauberian perturbation $U + K \in \mathcal{T}^d(X, Y)$, whenever $U \in \mathcal{T}^d(X, Y)$.

The sets of tauberian perturbation and co-tauberian perturbation are respectively denoted by $\mathcal{PT}(X, Y)$, and $\mathcal{PT}^d(X, Y)$. For $X = Y$, the previous sets are respectively denoted by $\mathcal{PT}(X)$, and $\mathcal{PT}^d(X)$.

Theorem 2.2. Let X be a Banach space. The $\mathcal{PT}(X)$ and $\mathcal{PT}^d(X)$ are two-sided ideals of $\mathcal{L}(X)$.

Proof. Let $T \in \mathcal{PT}(X)$ and $F \in \mathcal{PT}(X)$. First, we will prove that $T - F \in \mathcal{PT}(X)$. Let $S \in \mathcal{T}(X)$. Since $T \in \mathcal{PT}(X)$, then $T + S \in \mathcal{T}(X)$. It follows from the fact that $-F \in \mathcal{PT}(X)$ that $[(T - F) + S] \in \mathcal{T}(X)$ which implies that $T - F \in \mathcal{PT}(X)$. Hence, $\mathcal{PT}(X)$ is a subgroup of $\mathcal{L}(X)$. Now, take $T \in \mathcal{PT}(X)$ and $S \in \mathcal{L}(X)$, we have to prove that TS and ST are in $\mathcal{PT}(X)$. To do this, suppose that $T \in \mathcal{PT}(X)$ and let $S \in \mathcal{L}(X)$ and $A \in \mathcal{T}(X)$. If S is bijective, then $S^{-1}A \in \mathcal{T}(X)$, hence $A + ST = S(S^{-1}A + T) \in \mathcal{T}(X)$, thus $ST \in \mathcal{PT}(X)$. If S is not bijective, take $\lambda \in \rho(S)$ such that $\lambda \neq 0$. Let $A \in \mathcal{T}(X)$, then $(S - \lambda)((S - \lambda)^{-1}A + T) = A + (S - \lambda)T \in \mathcal{T}(X)$. It follows that $(S - \lambda)T \in \mathcal{PT}(X)$. Since $\lambda T \in \mathcal{PT}(X)$, then so is ST.

The proof of the second result is similar. □

Lemma 2.3. [10] Let $T \in \mathcal{L}(L_1(\mu), Y)$. If $T \in \mathcal{T}(L_1(\mu), Y)$ and $S \in \mathcal{L}(L_1(\mu), Y)$ satisfies $\|S\| < \beta_T$, then $T + S \in \mathcal{T}(L_1(\mu), Y)$.

Corollary 2.4. The class of $\mathcal{PT}(L_1(\mu), Y)$ is a closed subset of $\mathcal{L}(L_1(\mu), Y)$.

Proof. Let $S \in \mathcal{T}(L_1(\mu), Y)$ and $(T_n)_{n\in\mathbb{N}} \subset \mathcal{PT}(L_1(\mu), Y)$ such that $T_n \longrightarrow T$ as $n \longrightarrow \infty$ in $L_1(\mu)$. For a sufficiency large n and by using Lemma 2.3, we obtain $[S - (T_n - T)] \in \mathcal{T}(L_1(\mu), Y)$. Since $T_n \in \mathcal{PT}(L_1(\mu), Y)$, we deduce that $(S + T) \in \mathcal{T}(L_1(\mu), Y)$, which implies that $T \in \mathcal{PT}(L_1(\mu), Y)$. □

For $S, T \in \mathcal{L}(X, Y)$, we define the operator $(S, T) : X \to Y \times Y$ by $(S, T)x = (Sx, Tx)$, where $Y \times Y$ is equipped with the product norm $\|(y_1, y_2)\|_1 = \|y_1\| + \|y_2\|$.

According to previous notations, we introduce a new class of operators named $\mathcal{T}$-singular as an extension of the class of Φ-singular operators defined in [1].

Definition 2.5. Suppose that $\mathcal{T}(X, Y)$ is non-empty and let $T \in \mathcal{L}(X, Y)$. We say that T is $\mathcal{T}$-singular, $T \in \mathcal{TS}$, when for each $S \in \mathcal{L}(X, Y)$, $(S, T) \in \mathcal{T}$ implies $S \in \mathcal{T}$.

Corollary 2.6. [11] The class of $\mathcal{T}(L_1(\mu), Y)$ is an open set of $\mathcal{L}(L_1(\mu), Y)$.

Proposition 2.7. The set $\mathcal{TS}(L_1(\mu), Y)$ is closed in $\mathcal{L}(L_1(\mu), Y)$.

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{TS}(L_1(\mu), Y)$ such that $T_n \to T \in \mathcal{L}(L_1(\mu), Y)$. We will show that $T \in \mathcal{TS}(L_1(\mu), Y)$. To do this, let $S \in \mathcal{L}(L_1(\mu), Y)$ and $(S, T) \in \mathcal{T}(L_1(\mu), Y \times Y)$. It follows from the fact that $\|(S, T_n) - (S, T)\| = \|T_n - T\|$, that $(S, T_n) \to (S, T)$. Since $\mathcal{T}(L_1(\mu), Y \times Y)$ is an open set, by Corollary 2.6, then there exists a positive integer n_0 such that $(S, T_{n_0}) \in \mathcal{T}(L_1(\mu), Y \times Y)$. Then, $T_{n_0} \in \mathcal{TS}(L_1(\mu), Y)$ which implies that $S \in \mathcal{T}(L_1(\mu), Y)$. Thus $T \in \mathcal{TS}(L_1(\mu), Y)$. □

The following theorem shows that, when $Y \times Y \simeq Y$, the set of tauberian perturbation coincides with the set of $\mathcal{T}$-singular operators.

Theorem 2.8. Let X and Y be two Banach spaces. Suppose that $\mathcal{T}(X, Y) \neq \emptyset$. If $Y \times Y \simeq Y$, then $\mathcal{TS}(X, Y) = \mathcal{PT}(X, Y)$.

Proof. Let $U : Y \times Y \to Y$ be a bijective isomorphism and let $V, W \in \mathcal{L}(Y)$ such that for $(y_1, y_2) \in Y \times Y$ we have $U(y_1, y_2) = Vy_1 + Wy_2$. If $T \in \mathcal{PT}(X, Y)$, such that for each $S \in \mathcal{L}(X, Y)$, $(S, T) \in \mathcal{T}$ then we have $U(S, T) = VS + WT \in \mathcal{T}$. By Theorem 2.2, we obtain $WT \in \mathcal{PT}(X, Y)$. Since $S \in \mathcal{T}$, then so is VS and hence $T \in \mathcal{TS}(X, Y)$. □

As examples of spaces that satisfy the condition $Y \simeq Y \times Y$ in Theorem 2.8 we mention the spaces c_0 and l_p for $1 \leq p < \infty$.

Proposition 2.9. Let X and Y be two Banach spaces. Given $T \in \mathcal{L}(X, Y)$ and an (into) isomorphism $F \in \mathcal{L}(Y, Z)$. If $FT \in \mathcal{TS}(X, Z)$, then $T \in \mathcal{TS}(X, Y)$.

Proof. Let $T \in \mathcal{L}(X, Y)$ and let $F \in \mathcal{L}(X, Z)$ be an isomorphism into Z such that $FT \in \mathcal{TS}(X, Z)$. Consider $S \in \mathcal{L}(X, Y)$ and assume that $(S, T) \in \mathcal{T}(X, Y \times Y)$. Then, $(FS, FT) = (F \times F)(S, T) \in \mathcal{T}(X, Z \times Z)$. Since $FT \in \mathcal{TS}(X, Z)$ we obtain that $FS \in \mathcal{T}(X, Z)$. By using [[10], Proposition 2.1.3] we get $S \in \mathcal{T}(X, Y)$ and hence $T \in \mathcal{TS}(X, Y)$. □

Now, let us introduce the following definition.

Definition 2.10. Let X and Y be two Banach spaces. Suppose that $\mathcal{T}^d(X, Y) \neq \emptyset$ and let $T \in \mathcal{L}(X, Y)$. We say that T is $\mathcal{T}^d$-cosingular, $T \in \mathcal{T}^d\mathcal{CS}$, if for each $S \in \mathcal{L}(X, Y)$, $S \in \mathcal{T}^d$ whenever $(S, T) \in \mathcal{T}^d$.

We can now study the previous results for the class of $\mathcal{T}^d$-cosingular operators.

Let $S, T \in \mathcal{L}(X,Y)$. The operator $[S,T]$ is defined from $X \times X$ into Y by $[S,T](x_1,x_2) = Sx_1 + Tx_2$, where $X \times X$ is endowed with $\|(x_1,x_2)\|_\infty = \max\{\|y_1\|, \|y_2\|\}$.

Theorem 2.11. Let X and Y be two Banach spaces. Suppose that $\mathcal{T}^d(X,Y) \neq \emptyset$. If $X \times X \simeq X$, then $\mathcal{T}^d\mathcal{CS}(X,Y) = \mathcal{PT}^d(X,Y)$.

Proof. Let $U : X \times X \to X$ be a bijective isomorphism and let $V, W \in \mathcal{L}(X)$ such that for $(x_1,x_2) \in X \times X$ we have $U(x_1,x_2) = Vx_1 + Wx_2$. If $T \in \mathcal{PT}^d(X,Y)$, for each $S \in \mathcal{L}(X,Y)$ such that $(S,T) \in \mathcal{T}^d$ we have $U(S,T) = VS + WT \in \mathcal{T}^d$. By Theorem 2.2, we obtain that $WT \in \mathcal{PT}^d(X,Y)$. So $VS \in \mathcal{T}^d$ and hence by applying [[10], Proposition 3.1.4] we deduce that $S \in \mathcal{T}^d$. Hence $T \in \mathcal{T}^d\mathcal{CS}(X,Y)$. □

Proposition 2.12. Let X and Y be two Banach spaces and let $T \in \mathcal{L}(X,Y)$. Given an onto operator $R \in \mathcal{L}(Z,X)$. If $TR \in \mathcal{T}^d\mathcal{CS}(Z,Y)$, then $T \in \mathcal{T}^d\mathcal{CS}(X,Y)$.

Proof. Let $S \in \mathcal{L}(X,Y)$ such that $[S,T] \in \mathcal{T}^d(X \times X, Y)$, and let $R : Z \to X$ be an onto operator. Then, the operator $R \times R \in \mathcal{L}(Z \times Z, X \times X)$ defined by $(R \times R)(a,b) = (Ra, Rb)$ is onto. Thus, $[S,T](R \times R) = [SR, TR] \in \mathcal{T}^d(Z \times Z, Y)$. Since $TR \in \mathcal{T}^d\mathcal{CS}(Z,Y)$, we obtain $SR \in \mathcal{T}^d(Z,Y)$, and hence [[10], Proposition 3.1.4] leads to $S \in \mathcal{T}^d(X,Y)$. Therefore $T \in \mathcal{T}^d\mathcal{CS}(X,Y)$. □

3. Tauberian and co-tauberian perturbations on products Banach spaces

This section deals with some results on the tauberian and co-tauberian perturbations sets on product spaces. Our first result is the following:

Lemma 3.1. Let X and Y be two Banach spaces. Suppose that $\mathcal{PT}(X,Y) \neq \emptyset$ (resp. $\mathcal{PT}^d(X,Y) \neq \emptyset$) and let $T \in \mathcal{PT}(X,Y)$ (resp. $\mathcal{PT}^d(X,Y)$), $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. Then $BTA \in \mathcal{PT}(X,Y)$ (resp. $\mathcal{PT}^d(X,Y)$).

Proof. The proof can be immediately deduced by applying Theorem 2.2. □

Now, let X_i, X, Y_i and Y be Banach spaces (for $i = 1,2$). Every operator $T \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$ admits a matricial representation

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in \mathcal{L}(X_1,Y_1)$, $B \in \mathcal{L}(X_2,Y_1)$, $C \in \mathcal{L}(X_1,Y_2)$ and $D \in \mathcal{L}(X_2,Y_2)$. Moreover, for every $T \in \mathcal{L}(X_1 \times X_2, Y)$ we can write

$$T = (A,B),$$

with $A \in \mathcal{L}(X_1,Y)$, $B \in \mathcal{L}(X_2,Y)$. Finally, every operator $T \in \mathcal{L}(X, Y_1 \times Y_2)$ admits a the following representation

$$T = \begin{pmatrix} A \\ C \end{pmatrix},$$

or (A,C) for short, with $A \in \mathcal{L}(X,Y_1)$ and $C \in \mathcal{L}(X,Y_2)$.

Proposition 3.2. Let X, Y, X_i and Y_i be Banach spaces, for $i = 1,2$. Then, the following assertions valid.

(*i*) Suppose that $\mathcal{T}(X_1 \times X_2, Y_1 \times Y_2) \neq \emptyset$ and let $T \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$. Then, $T \in \mathcal{PT}$ if and only if $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ belong to $\mathcal{PT}$.

(*ii*) Assume $\mathcal{T}(X, Y_1 \times Y_2) \neq \emptyset$ and let $T \in \mathcal{L}(X, Y_1 \times Y_2)$. Then $T \in \mathcal{PT}$ if and only if $\begin{pmatrix} A \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ C \end{pmatrix}$ belong to $\mathcal{PT}$.

(iii) Assume $\mathcal{T}(X_1 \times X_2, Y) \neq \emptyset$ and let $T \in \mathcal{L}(X_1 \times X_2, Y)$. Then $T \in \mathcal{PT}$ if and only if $(A, 0)$ and $(B, 0)$ belong to $\mathcal{PT}$.

Proof. (i) To prove the direct implication, it is enough to see that $\mathcal{PT}(X_1 \times X_2, Y_1 \times Y_2)$ is a subspace of $\mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$. Conversely, since these matrices can be written respectively as P_1TQ_1, P_1TQ_2, P_2TQ_1 and P_2TQ_2, where P_i is the projection of $Y_1 \times Y_2$ onto Y_i, and Q_i is the projection of $X_1 \times X_2$ onto X_i, then by using Lemma 3.1, we achieve the desired result.

The proofs of (ii) and (iii) can be checked similarly. □

Proposition 3.3. Let X, Y, X_i and Y_i be Banach spaces, for $i = 1, 2$. Then, the three following assertions hold.

(i) Suppose that $\mathcal{T}^d(X_1 \times X_2, Y_1 \times Y_2) \neq \emptyset$ and let $T \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$. Then, $T \in \mathcal{PT}^d$ if and only if $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ belong to $\mathcal{PT}^d$.

(ii) Assume $\mathcal{T}^d(X, Y_1 \times Y_2) \neq \emptyset$ and let $T \in \mathcal{L}(X, Y_1 \times Y_2)$. Then $T \in \mathcal{PT}^d$ if and only if $\begin{pmatrix} A \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ C \end{pmatrix}$ belong to $\mathcal{PT}^d$.

(iii) Assume $\mathcal{T}^d(X_1 \times X_2, Y) \neq \emptyset$ and let $T \in \mathcal{L}(X_1 \times X_2, Y)$. Then $T \in \mathcal{PT}^d$ if and only if $(A, 0)$ and $(B, 0)$ belong to $\mathcal{PT}^d$.

Proof. The proof can be checked by a similar way as in the proof of Proposition 3.2. □

Remark 3.4. Let X, Y, X_i and Y_i be Banach spaces, for $i = 1, 2$. Then,

(i) An operator $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$ belongs to $\mathcal{T}$ if and only if both A and D belong to $\mathcal{T}$.

(ii) Suppose that both $\mathcal{T}(X_1, Y_1)$ and $\mathcal{T}(X_2, Y_2)$ are nonempty, then so is $\mathcal{T}(X_1 \times X_2, Y_1 \times Y_2)$.

(iii) An operator $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$ belongs to $\mathcal{T}^d$ if and only if both A and D belong to $\mathcal{T}^d$.

(iv) Suppose that both $\mathcal{T}^d(X_1, Y_1)$ and $\mathcal{T}^d(X_2, Y_2)$ are nonempty, then so is $\mathcal{T}^d(X_1 \times X_2, Y_1 \times Y_2)$.

We can also show that, if $T \in \mathcal{PT}(X_1 \times X_2, Y_1 \times Y_2)$ (resp. $\mathcal{PT}^d(X_1 \times X_2, Y_1 \times Y_2)$), then $A \in \mathcal{PT}(X_1, Y_1)$ (resp. $\mathcal{PT}^d(X_1, Y_1)$) and $D \in \mathcal{PT}(X_2, Y_2)$ (resp. $\mathcal{PT}^d(X_2, Y_2)$).

Proposition 3.5. Suppose that $\mathcal{T}(X_1, Y) \neq \emptyset \neq \mathcal{T}(X_2, Y)$ and $Y \simeq Y \times Y$. Let $(A, B) \in \mathcal{L}(X_1 \times X_2, Y)$. Then

$$(A, B) \in \mathcal{PT} \Rightarrow A \in \mathcal{PT}(X_1, Y) \text{ and } B \in \mathcal{PT}(X_2, Y).$$

Proof. It yields from Proposition 3.2 that both $(A, 0)$ and $(0, B)$ are in $\mathcal{PT}$. Given the assumption, there exist closed subspaces Y_1 and Y_2 of Y such that $Y_1 \simeq Y_2 \simeq Y = Y_1 \oplus Y_2$. Therefore, for each $i = 1, 2$, there exists an isomorphism $U_i \in \mathcal{L}(Y)$ with range $\mathcal{R}(U_i) = Y_i$. Additionally, $\mathcal{T}(X_1 \times X_2, Y) \neq \emptyset$. Let $T_i \in \mathcal{T}(X_i, Y)$ $(i = 1, 2)$. Then $(U_1T_1, U_2T_2) \in \mathcal{T}$. Since $(U_1A, 0) \in \mathcal{PT}$, we have $(U_1(T_1 + A), U_2T_2) \in \mathcal{T}$. This implies $U_1(T_1 + A) \in \mathcal{T}$. Therefore, by [[10], Proposition 2.1.3], we conclude that $T_1 + A \in \mathcal{T}(X_1, Y)$, and consequently, $A \in \mathcal{PT}$. Similarly, it can be shown that $B \in \mathcal{PT}$. □

Proposition 3.6. Suppose $\mathcal{T}(X, Y_1) \neq \emptyset \neq \mathcal{T}(X, Y_2)$ and let $(A, C) \in \mathcal{L}(X, Y_1 \times Y_2)$. Then

$$\begin{pmatrix} A \\ C \end{pmatrix} \in \mathcal{PT} \Rightarrow A \in \mathcal{PT}(X, Y_1) \text{ and } C \in \mathcal{PT}(X, Y_2).$$

Proof. Let $T \in \mathcal{T}(X, Y_1)$. This implies $(T, 0) \in \mathcal{T}(X, Y_1 \times Y_2)$. Using Proposition 3.2, we get $(A, 0) \in \mathcal{PT}$. Therefore, $(T + A, 0) \in \mathcal{T}$. Consequently, $T + A \in \mathcal{T}(X, Y_1)$, which implies that $A \in \mathcal{PT}$.

Similarly, we can check that $C \in \mathcal{PT}$. □

Proposition 3.7. Suppose that $\mathcal{T}^d(X, Y_1) \neq \emptyset \neq \mathcal{T}^d(X, Y_2)$ and $X \simeq X \times X$. Let $(A, C) \in \mathcal{L}(X, Y_1 \times Y_2)$. Then

$$(A, C) \in \mathcal{PT}^d(X, Y_1 \times Y_2) \Rightarrow A \in \mathcal{PT}^d(X, Y_1) \text{ and } C \in \mathcal{PT}^d(X, Y_2).$$

Proof. Suppose that $(A, C) \in \mathcal{PT}^d(X, Y_1 \times Y_2)$, we have

$$\begin{pmatrix} A \\ 0 \end{pmatrix} \in \mathcal{PT}^d(X, Y_1 \times Y_2) \text{ and } \begin{pmatrix} 0 \\ C \end{pmatrix} \in \mathcal{PT}^d(X, Y_1 \times Y_2),$$

by Proposition 3.2. Let $T_1 \in \mathcal{T}^d(X, Y_1)$ and $T_2 \in \mathcal{T}^d(X, Y_2)$. Given that X contains subspaces X_1 and X_2 isomorphic to X such that $X = X_1 \oplus X_2$, let $U_1 \in \mathcal{L}(X)$ be an operator that is bijective from X_1 onto X with kernel $\mathcal{N}(U_1) = X_2$, and let $U_2 \in \mathcal{L}(X)$ be an operator that is bijective from X_2 onto X with kernel $\mathcal{N}(U_2) = X_1$. We observe that

$$\begin{pmatrix} T_1U_1 \\ T_2U_2 \end{pmatrix} \in \mathcal{T}^d(X, Y_1 \times Y_2).$$

Using Proposition 3.2, we have

$$\begin{pmatrix} AU_1 \\ 0 \end{pmatrix} \in \mathcal{PT}^d(X, Y_1 \times Y_2).$$

Accordingly,

$$\begin{pmatrix} (T_1 + A)U_1 \\ T_2U_2 \end{pmatrix} \in \mathcal{T}^d(X, Y_1 \times Y_2),$$

which implies $(T_1+A)U_1 \in \mathcal{T}^d(X, Y_1)$. Therefore, according to [[10], Proposition 3.1.4], we have $T_1 + A \in \mathcal{T}^d(X, Y_1)$, proving that $A \in \mathcal{PT}^d(X, Y_1)$. Arguing by a similar reasoning, we show that $A \in \mathcal{PT}^d(X, Y_2)$. □

Proposition 3.8. Suppose $\mathcal{T}^d(X_1, Y) \neq \emptyset \neq \mathcal{T}^d(X_2, Y)$. If $(A, B) \in \mathcal{PT}^d(X_1 \times X_2, Y)$, then $A \in \mathcal{PT}^d(X_1, Y)$ and $B \in \mathcal{PT}^d(X_2, Y)$.

Proof. Consider $T \in \mathcal{T}^d(X_1, Y)$. This implies $(T, 0) \in \mathcal{T}^d(X_1 \times X_2, Y)$. Additionally, since $(A, B) \in \mathcal{PT}^d(X_1 \times X_2, Y)$, Proposition 3.2 implies that $(A, 0) \in \mathcal{PT}^d(X_1 \times X_2, Y)$. Therefore, $(T + A, 0) \in \mathcal{PT}^d(X_1 \times X_2, Y)$, which implies $T + A \in \mathcal{T}^d(X_1, Y)$. This proves that $A \in \mathcal{PT}^d(X_1, Y)$. Similarly, we can prove that $B \in \mathcal{PT}^d(X_2, Y)$. □

4. Some results on tauberian and co-tauberian theory of 2×2 operator matrices

In this section we establish some results on perturbation theory of 2×2 matrix operator that acts on a product of two Banach spaces which extend some results in [9].

Let X_1 and X_2 be two Banach spaces. In the product space $X_1 \times X_2$ we consider the matrix operator M_C as:

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{4.1}$$

where $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$ and $C \in \mathcal{L}(X_2, X_1)$.

Lemma 4.1. Let M_C be the operator defined in (4.1). Then, for every $C \in \mathcal{L}(X_2, X_1)$, the two following statements hold.
(i) If $A \in \mathcal{T}(X_1)$ and $B \in \mathcal{T}(X_2)$, then $M_C \in \mathcal{T}(X_1 \times X_2)$.
(ii) If $A \in \mathcal{T}^d(X_1)$ and $B \in \mathcal{T}^d(X_2)$, then $M_C \in \mathcal{T}^d(X_1 \times X_2)$.

Proof. (i) The operator M_C can be written in the following form

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{4.2}$$

Since $A \in \mathcal{T}(X_1)$ and $B \in \mathcal{T}(X_2)$, then it yields that $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$, $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$ are both tauberian operators. Taking into account that $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible, for every $C \in \mathcal{L}(X_1 \times X_2)$, then $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is tauberian. By using [[10], Proposition 2.1.3] we conclude that $M_C \in \mathcal{T}(X_1 \times X_2)$.
(ii) The proof of (ii) is established similarly as that in (i). □

Remark 4.2. Let X_1 and X_2 be two Banach spaces. Then, by using a similar reasoning as in again Lemma 4.1, it is easy to check that:
(i) If $A \in \mathcal{T}(X_1)$ and $B \in \mathcal{T}(X_2)$, then for every $D \in \mathcal{L}(X_1, X_2)$

$$M_D = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$$

is a tauberian operator on $X_1 \times X_2$.
If $A \in \mathcal{T}^d(X_1)$ and $B \in \mathcal{T}^d(X_2)$, then for every $D \in \mathcal{L}(X_1, X_2)$ M_D is a co-tauberian operator on $X_1 \times X_2$.

Lemma 4.3. Let X_1 and X_2 be two Banach spaces. Then, the following statements hold.

(i) If $M_C \in \mathcal{T}(X_1 \times X_2)$, then $A \in \mathcal{T}(X_1)$.

(ii) If $M_C \in \mathcal{T}^d(X_1 \times X_2)$, then $B \in \mathcal{T}^d(X_2)$.

Proof. The result follows immediately from Eq. (4.2). □

The following definition is an extension of the concept of g-Riesz operators [5].

Definition 4.4. Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is called a $\mathcal{T}$-Riesz operator if $(T - \lambda) \in \mathcal{T}(X, Y)$ for all scalars $\lambda \neq 0$.
The set of $\mathcal{T}$-Riesz operators from X to Y is denoted by $\mathcal{R}_{\mathcal{T}}(X, Y)$.

Note that, for $\lambda = 1$, $\mathcal{R}_{\mathcal{T}}(X, Y) = \mathcal{WDC}(X, Y)$.
As an example of $\mathcal{T}$-Riesz operators we cite the following.

Definition 4.5. [2] The Rademacher functions $(r_k)_{k=1}^{\infty}$ are defined on $[0, 1]$ by:

$$r_k(t) := sgn(\sin 2^k \pi t),$$

satisfy the following properties:
- $(r_k)(t) = \pm 1$ a.e for all k,
- $\int_0^1 r_{k_1} r_{k_2} \dots r_{k_m}(t)dt = 0$, whenever $k_1 < k_2 < \dots k_m$.

We denote by L_p the space $L_p(0,1)$ for $1 < p < \infty$. Let R_p be the closed subspace spanned in L_p, $1 < p < \infty$, by the Rademacher functions $(r_k)_{k=1}^{\infty}$ which is isomorphic to l_2. For more detail see [2].

Example 4.6. For $1 < p < \infty$, there exists a projection $P \in \mathcal{L}(L_p)$ into R_p. The operator $I - P : L_p \to L_p$ has a closed range, since $\mathcal{R}(I-P) = \mathcal{N}(P)$. Hence, by reflexivity of the space L_p, we deduce that $I - P \in \mathcal{T}(L_p)$.

Remark 4.7. Let X and Y be two Banach spaces. It is clear that

$$\mathcal{R}(X,Y) \subset \mathcal{R}_g(X,Y) \subset \mathcal{R}_{\mathcal{T}}(X,Y).$$

The following lemma concerns the class of $\mathcal{T}$-Riesz operators.

Lemma 4.8. Let X_1 and X_2 be two Banach spaces. Then, A and B are $\mathcal{T}$-Riesz if and only if, $\lambda - M_C$ is tauberian, for all $\lambda \in \mathbb{C}\backslash(\{0\} \cap \sigma(B))$.

Proof. We have

$$\lambda - M_C = \begin{pmatrix} \lambda - A & C \\ 0 & \lambda - B \end{pmatrix}.$$

Since A and B are $\mathcal{T}$-Riesz, then $\lambda - A$ and $\lambda - B$ are tauberian operators for all $\lambda \in \mathbb{C}\backslash\{0\}$, by Lemma 4.1, $(\lambda - M_C) \in \mathcal{T}(X_1 \times X_2)$. For the converse, let $\lambda \in \mathbb{C}\backslash\sigma(B)$, then we consider the following decomposition:

$$\lambda - M_C = \begin{pmatrix} \lambda - A & -C \\ 0 & \lambda - B \end{pmatrix} = \begin{pmatrix} I & -C(\lambda - B)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda - B \end{pmatrix}.$$

From the facts that $M_C \in \mathcal{R}_{\mathcal{T}}(X_1 \times X_2)$ and

$$\begin{pmatrix} \lambda - A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda - B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \lambda - B \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & I \end{pmatrix},$$

then by using Lemma 4.3, it follows that $\lambda - A$ and $\lambda - B$ are tauberian, which achieves the result. □

For $S \in \mathcal{L}(X,Y)$, we define the S-tauberian (resp., S-co-tauberian) spectrum of a bounded linear operator A as:

$$\begin{aligned} \sigma_{S,\mathcal{T}}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } (\lambda S - A) \notin \mathcal{T}(X,Y)\}, \\ \sigma_{S,\mathcal{T}^d}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } (\lambda S - A) \notin \mathcal{T}^d(X,Y)\}. \end{aligned}$$

Clearly, for $S = I$, we have the tauberian (resp., co-tauberian) spectra of A, which is designated by $\sigma_{\mathcal{T}}(A)$ (resp., $\sigma_{\mathcal{T}^d}(A)$).

Remark 4.9. Notice that the previous sets are bounded and not closed. However, when:

- $X = L_1(\mu)$, the tauberian spectrum is a closed compact set.
- $Y = L_\infty(\mu)$, the co-tauberian spectrum is a closed compact set,

where μ is a non-purely atomic, finite measure.

Theorem 4.10. Let X_1 and X_2 be two Banach spaces and let $A \in \mathcal{L}(X_1)$ and $B \in \mathcal{L}(X_2)$. Then for every $C \in \mathcal{L}(X_2, X_1)$ we have, then

$$\sigma_{\mathcal{T}}(M_C) \subset \sigma_{\mathcal{T}}(A) \cup \sigma_{\mathcal{T}}(B) \text{ and } \sigma_{\mathcal{T}^d}(M_C) \subset \sigma_{\mathcal{T}^d}(A) \cup \sigma_{\mathcal{T}^d}(B).$$

Proof. Let $\lambda \notin [\sigma_{\mathcal{T}}(A) \cup \sigma_{\mathcal{T}}(B)]$, then $\lambda - A \in \mathcal{T}(X_1)$ and $\lambda - B \in \mathcal{T}(X_2)$. It follows from assertion (i) of Lemma 4.1, that $\lambda - M_C \in \mathcal{T}(X_1 \times X_2)$ and therefore $\lambda \notin \sigma_{\mathcal{T}}(M_C)$. For the second inclusion, let $\lambda \notin [\sigma_{\mathcal{T}^d}(A) \cup \sigma_{\mathcal{T}^d}(B)]$, then $\lambda - A \in \mathcal{T}^d(X_1)$ and $\lambda - B \in \mathcal{T}^d(X_2)$. By assertion (ii) of Lemma 4.1, we deduce that $\lambda - M_C \in \mathcal{T}^d(X_1 \times X_2)$. Hence $\lambda \notin \sigma_{\mathcal{T}^d}(M_C)$. □

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