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LACUNARY CONVERGENCE OF SEQUENCES OF COMPLEX UNCERTAIN VALUED FUNCTIONS

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Abstract. This paper studies convergence concepts for sequences of complex uncertain valued functions. We introduce the lacunary summability function space and examine its relation to a Cesàro-type summability space, establishing conditions under which one implies the other. Further, we define and analyze four modes of lacunary strong convergence: almost surely, in measure, in mean, and in distribution. An inclusion hierarchy among these modes is established, and its strictness is demonstrated.

1. Introduction

Uncertainty theory introduced by Liu [13] in the year 2007 offers a mathematical framework for managing human-based uncertainty, and one of its core topics is the study of convergence in uncertain sequences. Further information on uncertainty theory and the convergence of uncertain sequences can be found in the works of You [22] and Chen et. al. [1]. The more work on uncertain sequences can be found in the literature, such as in [2, 3, 4, 16, 17, 18, 20, 21].

The initial work on lacunary sequences can be found in Freedman et al. [10]. Their study of the $|\sigma_1|$ class (originally introduced by Lorentz [14]) of strongly Cesàro summable sequences with a general lacunary sequence θ led to the definition of a larger class N_θ in the BK-spaces (Banach spaces of sequences in which the coordinate linear functionals are continuous). The convergence of sequences has been studied through various approaches in the literature, as seen in Maddox [15], King [12], Petersen [19]and Kamthan and Gupta [11]. Quite recently, Dowari and Triapthy [5, 6, 7, 8, 9] have studied the lacunary convergence of complex uncertain variables.

The sequences of uncertain functions have not been studied till date. Uncertain functions, in contrast to traditional deterministic or probabilistic function spaces, take values in the set of uncertain variables whose behavior is controlled by uncertainty measures and uncertainty distributions. In particular, we investigate the behavior of such sequences through the lens of strong lacunary convergence, which generalizes classical convergence by incorporating a lacunary index sequence. Leveraging the concept of the uncertainty density function, we define and explore four forms of strong lacunary convergence, namely lacunary strong convergence almost surely (LSCAS), in measure (LSCM) in mean (LSCMn), and in distribution (LSCD).

To highlight the distinctions between these convergence types, we construct explicit counterexamples demonstrating that none of the converses hold. The results provide deeper insight into the structure of convergence behavior within uncertain normed spaces, enriching the theory of uncertain functions.

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2. Definitions and Preliminaries

The uncertain measure is defined as follows, according to Liu [13].
Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ is called an *uncertain measure* if it satisfies the following axioms:

(1) (Normality) $\mathcal{M}\{\Gamma\} = 1$.
(2) (Duality) For any $\Lambda \in \mathcal{L}$, we have

$$\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1.$$

(3) (Subadditivity) For every countable sequence $\{\lambda_j\} \subset \mathcal{L}$, the measure satisfies

$$\mathcal{M}\left(\bigcup_{j=1}^{\infty} \lambda_j\right) \leq \sum_{j=1}^{\infty} \mathcal{M}\{\lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an *uncertainty space*, and each element $\Lambda \in \mathcal{L}$ is called an *event*.

To define the uncertain measure of a compound event, Liu [13] introduced the following product axiom:

(1) (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, 3, \ldots$. The *product uncertain measure* $\mathcal{M}$ is defined by

$$\mathcal{M}\left(\prod_{k=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k is an event in L_k for each k.

Below we define uncertain functions and different convergence concepts of sequences of uncertain functions:

Definition 2.1. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An **uncertain function** is a mapping*

$$f : \Gamma \to \mathbb{C}$$

that is $\mathcal{L}$-measurable. In particular, if $\xi : \Gamma \to \mathbb{R}$ (or $\mathbb{C}$) is an uncertain variable, then any function of ξ, such as

$$f(\xi) = 2\xi + 1, \quad g(\xi) = \xi^2, \quad h(\xi) = e^{\xi},$$

is an uncertain function, since it maps uncertain inputs to uncertain outputs.

Definition 2.2. *A sequence of uncertain functions $\{f_n(\gamma)\}$ is said to converge **almost surely** to $f(\gamma)$ if:*

$$\lim_{n\to\infty} f_n(\gamma) = f(\gamma)$$

for almost all γ in the uncertainty space.

Definition 2.3. *A sequence of uncertain functions $\{f_n(\gamma)\}$ **converges in measure** to $f(\gamma)$ if, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathcal{M}\left(\|f_n(\gamma) - f(\gamma)\| \geq \varepsilon\right) = 0,$$

where $\mathcal{M}$ is the uncertainty measure.

Definition 2.4. *For $p \geq 1$, a sequence of uncertain functions $\{f_n(\gamma)\}$ **converges in p-th mean** to $f(\gamma)$ if:*

$$\lim_{n\to\infty} \mathbb{E}\left[\|f_n(\gamma) - f(\gamma)\|^p\right] = 0.$$

*For $p = 1$, this is called **mean convergence**, and for $p = 2$, it is **mean square convergence**.*

Definition 2.5. *A sequence of uncertain functions* $\{f_n(\gamma)\}$ ***converges in distribution*** *to* $f(\gamma)$ *if the uncertain distribution functions satisfy:*

$$\lim_{n\to\infty} F_n(\gamma') = F(\gamma'), \quad \textit{for all continuity points of } F(\gamma'),$$

where $F_n(\gamma')$ *and* $F(\gamma')$ *are the uncertainty distribution functions of* $\{f_n(\gamma)\}$ *and* $f(\gamma)$*, respectively.*

3. Inclusion Relation of Different Convergence Concepts of Sequences of Uncertain Functions

In this section, we undertake a detailed study of various convergence concepts associated with sequences of complex uncertain-valued functions. Each mode of convergence captures a different perspective on how such functions approach their limit within the framework of an uncertain space. By analyzing these notions individually, we aim to highlight their distinctive features, explore their interrelations, and identify the circumstances under which one form of convergence implies another.

3.1. **Almost Sure Convergence Implies Convergence in Uncertainty Measure.**

Theorem 3.1. *If a sequence of uncertain functions* $\{f_n(\gamma)\}$ *converges almost surely to* $f(\gamma)$*, then it also converges in uncertainty measure.*

Proof. Almost sure convergence means,

$$\mathcal{M}\left(\lim_{n\to\infty} f_n(\gamma) - f(\gamma)\right) = 1, \quad \forall \gamma \in \Gamma. \tag{3.1}$$

This implies that for any fixed $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$,

$$\mathcal{M}(\|f_n(\gamma) - f(\gamma)\| \geq \varepsilon) = 0. \tag{3.2}$$

Since the measure of the set where $f_n(\gamma)$ is not close to $f(\gamma)$ goes to zero, we conclude:

$$\lim_{n\to\infty} \mathcal{M}(\|f_n(\gamma) - f(\gamma)\| \geq \varepsilon) = 0. \tag{3.3}$$

Thus, $f_n(\gamma)$ converges in uncertainty measure to $f(\gamma)$. □

The following examples illustrates the converse of the theorem.

Example 3.1. *Convergence in an uncertainty measure does not imply Almost Sure Convergence.*

Proof. Consider a sequence of uncertain functions $\{f_n(\gamma)\}$ defined as follows:

$$f_n(\gamma) = \begin{cases} 1, & \text{with measure } \frac{1}{n}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

First, we show the convergence in uncertainty measure.

For any fixed $\varepsilon > 0$, we compute the uncertainty measure:

$$\mathcal{M}(\|f_n(\gamma) - 0\| \geq \varepsilon) = \mathcal{M}(f_n(\gamma) = 1) = \frac{1}{n}. \tag{3.5}$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$, we conclude that:

$$\lim_{n\to\infty} \mathcal{M}(\|f_n(\gamma) - 0\| \geq \varepsilon) = 0. \tag{3.6}$$

Thus, $\{f_n(\gamma)\}$ **converges in the uncertainty measure** to 0.

Next, we show that it does not converge almost surely.

Almost sure convergence means that $f_n(\gamma)$ must be eventually equal to 0 for all γ, except on a set of measure zero.

However, here the measure of $f_n(\gamma) = 1$ is $\frac{1}{n}$, which means that for any fixed γ, the event $f_n(\gamma) = 1$ occurs **infinitely often** with positive measure.

Since $\{f_n(\gamma)\}$ does not become permanently zero with measure 1, it does **not** converge almost surely. □

3.2. **Mean Convergence Implies Convergence in Uncertainty Measure.**

Theorem 3.2. *If $\{f_n(\gamma)\}$ converges in the mean, then it also converges in uncertainty measure.*

Proof. Mean convergence means:

$$\lim_{n\to\infty} \mathbb{E}[\|f_n(\gamma) - f(\gamma)\|] = 0. \tag{3.7}$$

By Markov's inequality, for any $\varepsilon > 0$:

$$\mathcal{M}(\|f_n(\gamma) - f(\gamma)\| \geq \varepsilon) \leq \frac{\mathbb{E}[\|f_n(\gamma) - f(\gamma)\|]}{\varepsilon}. \tag{3.8}$$

Since $\mathbb{E}[\|f_n(\gamma) - f(\gamma)\|] \to 0$, the right-hand side approaches 0 as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \mathcal{M}(\|f_n(\gamma) - f(\gamma)\| \geq \varepsilon) = 0. \tag{3.9}$$

which shows that $\{f_n(\gamma)\}$ converges in uncertainty measure. □

Example 3.2. *Suppose that $f_n(\gamma) = \frac{1}{n}\gamma$, where γ is an uncertain variable. Since $\mathbb{E}[\|f_n(\gamma)\|] = \frac{1}{n}\mathbb{E}[\|\gamma\|]$, which goes to zero, we conclude that $f_n(\gamma)$ converges in uncertainty measure.*

Example 3.3. *Convergence in an uncertainty measure does not imply convergence in mean.*

Proof. Consider $\{f_n(\gamma)\}$ defined as:

$$f_n(\gamma) = \begin{cases} n, & \text{with measure } \frac{1}{n^2}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

For any fixed $\varepsilon > 0$, we have:

$$\mathcal{M}(\|f_n(\gamma) - 0\| \geq \varepsilon) = \frac{1}{n^2} \to 0, \quad \text{as } n \to \infty. \tag{3.11}$$

Thus, $f_n(\gamma)$ converges in uncertainty measure to 0. However, its expected value is:

$$\mathbb{E}[\|f_n(\gamma)\|] = n \cdot \frac{1}{n^2} = \frac{1}{n}. \tag{3.12}$$

Since $\frac{1}{n}$ does not necessarily tend to zero in an absolute sense, this shows that convergence in the uncertainty measure does not imply mean convergence.

□

4. The Space N_θ for Uncertain Functions

In this section, we extend lacunary summability and strong almost convergence to sequences of uncertain functions within an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, using the uncertainty norm $\|\cdot\|_\Gamma$. The lacunary summability space $N_\theta[U, f]$ is defined via index gaps, with convergence governed by the growth rate $q_r = k_r/k_{r-1}$. Strong almost convergence, a refinement of Cesàro convergence, is shown to be strictly contained in $N_\theta[U, f]$, thereby establishing a hierarchy of convergence concepts for uncertain functions.

Definition 4.1 (Strong Cesàro Summability Space, $|\sigma_1|_f^U$)**.** *A sequence $\{f_k(\gamma)\}$ of uncertain functions is said to be* strongly Cesàro summable *to an uncertain function $g(\gamma)$ if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|f_k(\gamma) - g(\gamma)\|_\Gamma = 0,$$

where $\|\cdot\|_\Gamma$ denotes a norm defined over the uncertainty space Γ, such as an expected value norm, supremum norm or other suitable functional depending on the structure of $(\Gamma, \mathcal{L}, \mathcal{M})$.

Remark 4.1. *This generalizes the classical notion of Cesàro summability for scalar sequences to the setting of uncertain functions, where convergence is measured with respect to uncertainty rather than over a fixed deterministic interval.*

Definition 4.2. *(Lacunary Summability Space, $N_\theta[U, f]$). Let $\theta = (k_r)$ be a lacunary sequence with $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$. Define $I_r = (k_{r-1}, k_r]$. For a sequence of uncertain functions $\{f_k(\gamma)\}$ on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with norm $\|\cdot\|_\Gamma$, we say $\{f_k(\gamma)\}$ belongs to $N_\theta[U, f]$ if there exists $g(\gamma)$ such that*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|f_k(\gamma) - g(\gamma)\|_\Gamma = 0.$$

In this case, $\{f_k(\gamma)\}$ is lacunary summable *to $g(\gamma)$ with respect to θ.*

Below, we discuss the inclusion results for *Cesàro and Lacunary Summability* under uncertainty.

Theorem 4.1. *If $\{f_k(\gamma)\} \in |\sigma_1|_f^U$, then $\{f_k(\gamma)\} \in N_\theta[U, f]$; hence*

$$|\sigma_1|_f^U \subset N_\theta[U, f].$$

Proof. Let $\{f_k(\gamma)\} \in |\sigma_1|_f^U$. Then by definition, there exists an uncertain function $g(\gamma)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|f_k(\gamma) - g(\gamma)\|_\Gamma = 0.$$

We must show that

$$\tau_r := \frac{1}{h_r} \sum_{k\in I_r} \|f_k(\gamma) - g(\gamma)\|_\Gamma \to 0 \quad \text{as } r \to \infty.$$

Since Cesàro means converge to zero, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\frac{1}{n} \sum_{k=1}^{n} \|f_k(\gamma) - g(\gamma)\|_\Gamma < \varepsilon.$$

Now take r large enough so that $k_{r-1} \geq N$. Then the block $I_r \subset \{k : k > N\}$, and so all the terms in the block average are small:

$$\tau_r = \frac{1}{h_r} \sum_{k\in I_r} \|f_k(\gamma) - g(\gamma)\|_\Gamma < \varepsilon.$$

Thus, $\tau_r \to 0$ as $r \to \infty$, showing that $\{f_k(\gamma)\} \in N_\theta[U, f]$. □

Theorem 4.2. *Let $\theta = (k_r)$ be a lacunary sequence with $q_r = k_r/k_{r-1}$ and $\liminf_r q_r > 1$. If a sequence $\{f_k(\gamma)\}$ satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|f_k(\gamma) - g(\gamma)\|_\Gamma = 0$$

for some $g(\gamma)$, then $\{f_k(\gamma)\} \in N_\theta[U, f]$, i.e., $|\sigma_1|_f^U \subset N_\theta[U, f]$.

Proof. Let $\varepsilon > 0$. Since the Cesàro means converge in the uncertainty norm, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\frac{1}{n}\sum_{k=1}^{n} \|f_k(\gamma) - g(\gamma)\|_\Gamma < \varepsilon.$$

For large r, since $h_r = k_r - k_{r-1} \to \infty$, we can write:

$$\frac{1}{h_r}\sum_{k\in I_r} \|f_k(\gamma) - g(\gamma)\|_\Gamma \leq \frac{1}{h_r}\left(\sum_{k=1}^{k_r} \|f_k - g(\gamma)\|_\Gamma - \sum_{k=1}^{k_{r-1}} \|f_k - g(\gamma)\|_\Gamma\right).$$

This difference is small because both Cesàro averages converge in the uncertainty norm. Hence the lacunary mean tends to zero, implying $\{f_k(\gamma)\} \in N_\theta[U, f]$. □

Theorem 4.3. *Let $\{f_k(\gamma)\}$ be a sequence in $N_\theta[U, f]$. Suppose there exists a constant $C > 0$ such that*

$$q_r = \frac{k_r}{k_{r-1}} \leq C \quad \textit{for all } r.$$

Then $\{f_k(\gamma)\} \in |\sigma_1|_f^U$, i.e., $N_\theta[U, f] \subset |\sigma_1|_f^U$.

Proof. We express the Cesàro mean as a weighted average of lacunary blocks:

$$\frac{1}{k_r}\sum_{k=1}^{k_r} \|f_k(\gamma) - g(\gamma)\|_\Gamma = \sum_{j=1}^{r} \frac{h_j}{k_r} \cdot \left(\frac{1}{h_j}\sum_{k\in I_j} \|f_k(\gamma) - g(\gamma)\|_\Gamma\right).$$

Since $\{f_k(\gamma)\} \in N_\theta[U, f]$, each average tends to zero. And because q_r is uniformly bounded, the weights $\frac{h_j}{k_r}$ are summable and bounded, preserving convergence of the Cesàro mean.

This shows $\{f_k(\gamma)\} \in |\sigma_1|_f^U$.

□

Theorem 4.4. *Let $\{f_k(\gamma)\}$ be a sequence of uncertain functions defined over the uncertainty measure space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then $N_\theta[U, f] \subseteq |\sigma_1|_f^U$ if and only if $\limsup_r q_r < \infty$, where $q_r = \frac{k_r}{k_{r-1}}$.*

Proof. **(Sufficiency)** Suppose $\limsup_r q_r < \infty$. Then there exists a constant $C > 0$ such that $q_r < C$ for all r. This implies the lacunary intervals do not increase too rapidly.

Let $\{f_k(\gamma)\} \in N_\theta^0[U, f]$, i.e., for every $\varepsilon > 0$, there exists R such that for all $r > R$,

$$\tau_r := \frac{1}{h_r}\sum_{k\in I_r} \|f_k(\gamma)\|_\Gamma < \varepsilon.$$

Now we consider the Cesàro means of the uncertainty norms:

$$A_t := \frac{1}{t}\sum_{k=1}^{t} \|f_k(\gamma)\|_\Gamma.$$

We take $t \in (k_{r-1}, k_r]$. Splitting the sum we get,

$$\sum_{k=1}^{t} \|f_k(\gamma)\|_\Gamma = \sum_{j=1}^{r-1}\sum_{k\in I_j} \|f_k(\gamma)\|_\Gamma + \sum_{k=k_{r-1}+1}^{t} \|f_k(\gamma)\|_\Gamma.$$

Using the bounds $\tau_j < \varepsilon$ for $j > R$, we estimate:

$$A_t \leq \frac{1}{t}\left(\sum_{j=1}^{R} h_j\tau_j + \sum_{j=R+1}^{r-1} h_j\varepsilon + h_r\varepsilon\right).$$

Here we have, $\sum_{j=1}^{R} h_j\tau_j$ is constant, $\sum_{j=R+1}^{r-1} h_j \leq k_{r-1} - k_R$, - $h_r < k_r - k_{r-1} < (C-1)k_{r-1}$.

Thus,

$$A_t \leq \frac{K}{t} + \varepsilon \cdot \frac{k_{r-1} - k_R + h_r}{t} \leq \frac{K}{t} + \varepsilon \cdot \frac{C \cdot k_{r-1}}{t}.$$

Since $k_{r-1} < t$, we get:

$$A_t < \frac{K}{t} + C\varepsilon.$$

Letting $t \to \infty$, we get $\limsup A_t \leq C\varepsilon$, and since ε is arbitrary, $A_t \to 0$. Hence, $\{f_k(\gamma)\} \in |\sigma_1|_f^U$.

(Necessity) Suppose $\limsup q_r = \infty$. Then for any j, we can find r_j with $q_{r_j} > j$. We define a sequence $\{f_k(\gamma)\}$ as:

$$f_k(\gamma) = \begin{cases} u(\gamma), & \text{if } k_{r_{j-1}} < k \leq 2k_{r_j-1}, \\ 0, & \text{otherwise,} \end{cases}$$

where $u(\gamma)$ is an uncertain function with $\|u(\gamma)\|_\Gamma = 1$, and $\mathcal{M}(u(\gamma)) = 1$.

Within the interval I_{r_j}, the number of non-zero terms is at most k_{r_j-1}, so:

$$\tau_{r_j} = \frac{1}{h_{r_j}} \sum_{k \in I_{r_j}} \|f_k(\gamma)\|_\Gamma \leq \frac{k_{r_j-1}}{k_{r_j} - k_{r_{j-1}}} < \frac{1}{j-1} \to 0.$$

Thus, $\{f_k(\gamma)\} \in N_\theta^0$.

Now we check for Cesàro summability. Over $[1, k_{r_j}]$, we have:

$$\frac{1}{k_{r_j}} \sum_{k=1}^{k_{r_j}} \|f_k(\gamma) - u(\gamma)\|_\Gamma \geq \frac{k_{r_j} - 2k_{r_j-1}}{k_{r_j}} = 1 - \frac{2k_{r_j-1}}{k_{r_j}} > 1 - \frac{2}{j}.$$

So the Cesàro mean does not converge to 0. Hence, $\{f_k(\gamma)\} \notin |\sigma_1|^0$, proving that the inclusion $N_\theta[U, f] \subseteq |\sigma_1|_f^U$ fails without bounded q_r. □

5. Summability Concepts via Uncertainty Density Function

In this section we introduce the uncertainty density function $\mu(\gamma)$ to refine summability and convergence in uncertainty normed spaces. Using this concept, we generalize the strong almost convergence and lacunary summability to the uncertain framework. We showed that while strongly almost convergent sequences are lacunary summable, the converse is not true, marking a strict inclusion.

Definition 5.1 (Uncertainty Density Function)**.** *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertainty density function $\mu : \Gamma \to [0, 1]$ represents the density of uncertainty over Γ, satisfying:*

(1) $\mu(\gamma) \geq 0$ *for all* $\gamma \in \Gamma$,
(2) $\int_\Gamma \mu(\gamma)\, d\gamma = 1$,

This function allows weighted evaluation of norms and convergence across the space of uncertainty.

Definition 5.2 (Uncertainty Density Norm)**.** *The uncertainty norm of an uncertain function $f(x, \gamma)$ at a fixed $x \in D$ is defined by*

$$\|f(x)\|_\rho := \int_\Gamma |f(x, \gamma)| \cdot \rho(\gamma)\, d\gamma.$$

Definition 5.3 (Strong Almost Convergence with Uncertainty Density)**.** *A sequence of uncertain functions $\{f_k(\gamma)\}$ is said to be strongly almost convergent to an uncertain function $l(\gamma) \in (\Gamma, \mathcal{L}, \mathcal{M})$ if*

$$\lim_{n\to\infty} \sup_{m\geq 0} \int_\Gamma \left| \frac{1}{n} \sum_{i=m+1}^{m+n} \|f_i(\gamma) - l(\gamma)\| \right| \mu(\gamma)\, d\gamma = 0.$$

This convergence considers the distribution of uncertainty via $\mu(\gamma)$ and we denote this space by AC_μ^U.

Definition 5.4. *Let $\{f_k\}$ be a sequence of complex uncertain functions defined on a domain $D \subseteq \mathbb{R}$, and let each $f_k : D \times \Gamma \to \mathbb{C}$. The sequence $\{f_k\}$ is said to be lacunary strongly convergent in uncertainty norm to an uncertain function f, if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|f_k(x) - f(x)\|_\mu = 0, \quad \text{for every } x \in D,$$

where for each $x \in D$, the uncertainty norm is given by

$$\|f_k(x) - f(x)\|_\mu = \int_\Gamma |f_k(x,\gamma) - f(x,\gamma)|\, d\mu(\gamma),$$

and $\{I_r\} = (k_{r-1}, k_r]$ is a lacunary sequence with $h_r = k_r - k_{r-1}$.
We denote this space by $N_\theta[U, f, \mu]$.

Definition 5.5. *The sequence $\{f_k\}$ is said to be lacunary strongly almost convergent to f if for every $\varepsilon > 0$, there exists $\Lambda \subseteq \Gamma$ with $\mathcal{M}(\Lambda) = 1$ such that*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|f_k(\gamma) - f(\gamma)\| = 0, \quad \text{for all } \gamma \in \Lambda.$$

Definition 5.6. *The sequence $\{f_k\}$ is said to be lacunary strongly convergent in measure to f if*

$$\lim_{r\to\infty} \mathcal{M}\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|f_k(\gamma) - f(\gamma)\| > \varepsilon \right\} = 0, \quad \forall \varepsilon > 0.$$

Definition 5.7. *The sequence $\{f_k\}$ is said to be lacunary strongly convergent in mean to f if*

$$\lim_{r\to\infty} \int_\Gamma \frac{1}{h_r} \sum_{k\in I_r} \|f_k(\gamma) - f(\gamma)\|\, \rho(\gamma)\, d\gamma = 0.$$

Definition 5.8. *Let Φ_k and Φ be the uncertainty distributions of f_k and f, respectively. Then $\{f_k\}$ is lacunary strongly convergent in distribution to f if for all continuity points $c \in \mathbb{C}$ of Φ,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|\Phi_k(c) - \Phi(c)\| = 0.$$

5.1. **Inclusion Result and Its Strictness.**

Theorem 5.1. *Let AC_μ^U be the space of sequences strongly almost convergent with respect to the uncertainty density. Then:*

$$AC_\mu \subsetneq N_\theta[U, f, \mu].$$

Proof. Suppose $\{f_k(\gamma)\} \in AC_\mu^U$. For every $\varepsilon > 0$, there exists $N > 0$ and $l(\gamma)$ such that

$$\sup_{m\geq 0} \int_\Gamma \left| \frac{1}{n} \sum_{i=m+1}^{m+n} \|f_i(\gamma) - l(\gamma)\| \right| \mu(\gamma)\, d\gamma < \varepsilon \quad \text{for all } n > N.$$

Since $h_r \to \infty$, for large enough r, $h_r > N$, hence the lacunary average also tends to zero:

$$\int_\Gamma \left| \frac{1}{h_r} \sum_{k \in I_r} \|f_k(\gamma) - l(\gamma)\| \right| \mu(\gamma)\, d\gamma < \varepsilon.$$

Thus, $\{f_k(\gamma)\} \in N_\theta[U, f, \mu]$.

To show the inclusion is strict, define a sequence:

$$f_k(\gamma) = \begin{cases} 1, & \text{if } k \in (k_{r-1}, k_{r-1} + \sqrt{h_r}] \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$\int_\Gamma \left(\frac{1}{h_r} \sum_{k \in I_r} \|f_k(\gamma)\| \right) \mu(\gamma)\, d\gamma = \frac{\sqrt{h_r}}{h_r} \int_\Gamma \mu(\gamma)\, d\gamma = \frac{1}{\sqrt{h_r}} \to 0.$$

Hence, $\{f_k(\gamma)\} \in N_\theta[U, f, \mu]$.

However, this sequence contains long blocks of 1's and 0's, causing the moving average

$$\frac{1}{n} \sum_{i=m+1}^{m+n} f_i(\gamma)$$

to oscillate in γ, and thus not converge uniformly in density. Therefore, $\{f_k(\gamma)\} \notin AC_\mu^U$. □

Now we discuss about the following inclusions:

$$\textbf{LSCAS} \Rightarrow \textbf{LSCM} \Rightarrow \textbf{LSCMn} \Rightarrow \textbf{LSCD}$$

1. LSCAS ⇒ LSCM.

Theorem 5.2. *If $\{f_k\}$ is lacunary strongly convergent almost surely to f, then it is also lacunary strongly convergent in measure to f.*

Proof. Assume,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \|f_k(x, \gamma) - f(x, \gamma)\| = 0 \quad \text{for a.e. } \gamma \in \Gamma.$$

For any $\varepsilon > 0$, define the set:

$$A_r^\varepsilon = \left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|f_k(x, \gamma) - f(x, \gamma)\| > \varepsilon \right\}$$

Since the convergence holds almost surely, by Egorov's Theorem,

$$\lim_{r \to \infty} \mathcal{M}(A_r^\varepsilon) = 0$$

Hence, $f_k \to f$ in lacunary strong convergence in measure. □

2. LSCM ⇒ LSCMn.

Theorem 5.3. *Lacunary strong convergence in measure implies lacunary strong convergence in mean.*

Proof. From LSCM, for every $\varepsilon > 0$,

$$\lim_{r \to \infty} \mathcal{M} \left(\gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|f_k(x, \gamma) - f(x, \gamma)\| > \varepsilon \right) = 0.$$

Using the inequality,

$$\int_{\Gamma}\left(\frac{1}{h_r}\sum_{k\in I_r}\|f_k(x,\gamma)-f(x,\gamma)\|\right)f(\gamma)\,d\gamma\leq\varepsilon+\mathcal{M}(A_r^{\varepsilon})\cdot C$$

As $\mathcal{M}(A_r^{\varepsilon})\to 0$, we get,

$$\lim_{r\to\infty}\int_{\Gamma}\frac{1}{h_r}\sum_{k\in I_r}\|f_k(x,\gamma)-f(x,\gamma)\|f(\gamma)\,d\gamma=0$$

Thus, LSCMn holds. □

3. LSCMn ⇒ LSCD.

Theorem 5.4. *Lacunary strong convergence in mean implies lacunary strong convergence in distribution.*

Proof. Let Φ_k and Φ be the uncertainty distributions of $f_k(x)$ and $f(x)$, respectively. From LSCMn:

$$\lim_{r\to\infty}\int_{\Gamma}\frac{1}{h_r}\sum_{k\in I_r}\|f_k(x,\gamma)-f(x,\gamma)\|f(\gamma)\,d\gamma=0$$

Then, using convergence in expectation implies convergence in distribution,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\Phi_k(c)-\Phi(c)\|=0$$

Hence, LSCD holds. □

Each implication is strict—i.e., none of the converses hold in general.

Counterexamples for Strictness

Example 5.1. ***LSCM does not imply LSCAS.***

Proof. Consider the uncertainty space $(\Gamma, L, \mathcal{M})$ with $\Gamma=[0,1]$ and uncertainty density $\mu(\gamma)=1$, i.e., uniform uncertainty.

We define a sequence of uncertain functions:

$$f_k(\gamma)=\begin{cases}1, & \text{if } \gamma\in\left[\frac{1}{2}-\frac{1}{2k},\frac{1}{2}+\frac{1}{2k}\right],\\ 0, & \text{otherwise.}\end{cases}$$

Let $f(\gamma)=0$, and let the lacunary sequence $\theta=\{k_r\}$ be given by $k_r=2^r$. Then $I_r=(k_{r-1},k_r]$, and $h_r=k_r-k_{r-1}\to\infty$.

Now,

For each k, the function $f_k(\gamma)$ is equal to 1 on an interval of length $\frac{1}{k}$, centered at $\gamma=\frac{1}{2}$, and zero elsewhere. So for fixed $\gamma\neq\frac{1}{2}$, $f_k(\gamma)=0$ for all sufficiently large k, and

$$\|f_k(\gamma)-f(\gamma)\|=0\quad\text{eventually.}$$

But for $\gamma=\frac{1}{2}$, we have $f_k(\gamma)=1$ for all k, so,

$$\|f_k(\gamma)-f(\gamma)\|=1\quad\text{for all } k.$$

Hence, the pointwise lacunary mean converges to 0 for almost all $\gamma\in\Gamma\setminus\{1/2\}$, but the convergence is not almost sure, since the set where convergence fails (i.e., $\{1/2\}$) has measure zero, but the convergence is not uniform outside a null set (as Egorov's condition for LSCAS fails due to the sharp oscillation).

However, in the measure sense, we obtain,

$$\int_{\Gamma}\left(\frac{1}{h_r}\sum_{k\in I_r}\|f_k(\gamma)\|\right)d\gamma\leq\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{k}\to 0.$$

Thus, $\{f_k\} \in \text{LSCM}$, but $\{f_k\} \notin \text{LSCAS}$. This shows that lacunary strong convergence in measure does not imply lacunary strong convergence almost surely. □

Example 5.2. ***LSCMn does not imply LSCM.***

Proof. Let $\Gamma = [0,1]$ and we take the uncertainty density $\mu(\gamma) = 1$. We define a sequence of uncertain functions $\{f_k(\gamma)\}$ by:

$$f_k(\gamma) = \begin{cases} k, & \text{if } \gamma \in \left[0, \frac{1}{k^2}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Let $f(\gamma) = 0$ for all $\gamma \in \Gamma$.

Let the lacunary sequence $\theta = \{k_r\}$ with $k_r = 2^r$ so that $h_r = k_r - k_{r-1} \to \infty$ and $I_r = (k_{r-1}, k_r]$.

First we show LSCMn holds.

We find the lacunary mean in the μ-integrated sense:

$$\int_\Gamma \left(\frac{1}{h_r} \sum_{k \in I_r} \|f_k(\gamma)\| \right) \mu(\gamma)\, d\gamma = \frac{1}{h_r} \sum_{k \in I_r} \int_0^{1/k^2} k\, d\gamma = \frac{1}{h_r} \sum_{k \in I_r} k \cdot \frac{1}{k^2} = \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{k} \to 0.$$

Hence, the sequence is lacunary strongly convergent in mean.

Next we show LSCM fails.

For any $\varepsilon = 1$, define:

$$A_k^\varepsilon = \{\gamma \in \Gamma : \|f_k(\gamma) - f(\gamma)\| > 1\} = \left[0, \frac{1}{k^2}\right].$$

Then:

$$\mathcal{M}(A_k^\varepsilon) = \int_0^{1/k^2} \mu(\gamma)\, d\gamma = \frac{1}{k^2}.$$

Now fix any r, then for $k \in I_r$, we have:

$$\mathcal{M}\left(\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|f_k(\gamma) - f(\gamma)\| > \varepsilon \right\} \right) \geq \mathcal{M}\left(\left[0, \frac{1}{k_{r-1}^2}\right] \right) = \frac{1}{k_{r-1}^2}.$$

This lower bound does not converge to 0 as $r \to \infty$. Hence, LSCM fails.

This sequence satisfies LSCMn but not LSCM, showing the inclusion is strict. □

Example 5.3. ***LSCD does not imply LSCMn.***

Proof. Let the uncertainty space be $(\Gamma, L, \mathcal{M})$, where $\Gamma = [0,1]$ and $\mu(\gamma) = 1$, i.e., uniform uncertainty.

We define the sequence of uncertain functions:

$$f_k(\gamma) = \begin{cases} k, & \text{if } \gamma \in \left[0, \frac{1}{k}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Let $f(\gamma) = 0$ for all $\gamma \in \Gamma$, and let the lacunary sequence $\theta = \{k_r\}$ with $k_r = 2^r$, so $h_r = k_r - k_{r-1} \to \infty$.

First we show LSCD holds.

For any bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we have:

$$\int_\Gamma \varphi(f_k(\gamma))\, d\gamma = \int_0^{1/k} \varphi(k)\, d\gamma + \int_{1/k}^1 \varphi(0)\, d\gamma = \varphi(k) \cdot \frac{1}{k} + \varphi(0) \cdot \left(1 - \frac{1}{k}\right).$$

As $k \to \infty$, $\varphi(k) \cdot \frac{1}{k} \to 0$, and we get:

$$\int_\Gamma \varphi(f_k(\gamma))\, d\gamma \to \varphi(0).$$

So the distribution of $f_k(\gamma)$ converges weakly to a point mass at 0, i.e., $\Phi_k \to \Phi$, and lacunary averaging over f_k yields convergence in distribution. Hence, $\{f_k\} \in \text{LSCD}$.

Here we show LSCMn fails.

We find the lacunary mean of the norms:

$$\int_\Gamma \left(\frac{1}{h_r} \sum_{k \in I_r} \|f_k(\gamma)\| \right) \mu(\gamma)\, d\gamma = \frac{1}{h_r} \sum_{k \in I_r} \int_0^{1/k} k\, d\gamma = \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{1} = 1.$$

So for each r, the mean value remains constant at 1. Thus,

$$\lim_{r \to \infty} \int_\Gamma \left(\frac{1}{h_r} \sum_{k \in I_r} \|f_k(\gamma) - f(\gamma)\| \right) \mu(\gamma)\, d\gamma = 1 \neq 0.$$

Hence, $\{f_k\} \notin \text{LSCMn}$.

This shows that convergence in distribution (LSCD) does not imply convergence in mean (LSCMn). □

6. Conclusion

In this paper, we studied convergence of complex uncertain-valued function sequences under an uncertainty norm. We defined the lacunary summability space $N_\theta[U, f]$ and related it to the Cesàro-type space $|\sigma_1|_f^U$. We further introduced four modes of lacunary strong convergence—almost surely, in measure, in mean, and in distribution—established their inclusion hierarchy and showed strictness with examples. These results provide a basis for further study of summability and convergence in uncertain settings.

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