Methods of Functional Analysis and Topology
Vol. 32 (2026), no. 3, pp. 254–265
doi.org/10.31392/MFAT-npu26_3.2026.06

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

WEIGHTED p-LAPLACIAN PROBLEMS WITH SINGULAR NONLINEARITIES HAVING VARIABLE EXPONENT

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Abstract. In this article we give an existence result of a weak solution for a weighted p-Laplacian problems with singular nonlinearities having variable exponent. The problem that we consider is the following

$$\begin{cases} u - \Delta_p^\nu u = \frac{h(x)}{u^{\gamma(x)}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Where p is a real number such that $1 < p < \infty$, $\gamma(x)$ is a continuous function on $\bar{\Omega}$ such that $1 \leq \gamma(x) \leq \gamma^+$, $\gamma^+ = \sup_{x\in\bar{\Omega}} \gamma(x)$, h is a nonnegative function belonging to the Lebesgue space $L^m(\Omega)$ and $m \geq 1$.

1. Introduction

The main goal of this article is to show the existence of a weak solution for a quasilinear elliptic problem with singular nonlinearities having variable exponent for suitable m and $\gamma(x)$. The problem that we consider is the following

$$\begin{cases} u - \Delta_p^\nu u = \frac{h(x)}{u^{\gamma(x)}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Where $\gamma(x)$ is a continuous function on $\bar{\Omega}$ with $1 \leq \gamma(x) \leq \gamma^+$, $\gamma^+ = \sup_{x\in\bar{\Omega}} \gamma(x)$, h is a nonnegative function belonging to the Lebesgue space $L^m(\Omega)$, p is a real number such that $1 < p < \infty$ and Ω is a bounded domain in $\mathbb{R}^N (N \geq p)$ with smooth boundary $\partial\Omega$. The well-known operator $\Delta_p^\nu u = \operatorname{div}\left(\nu|\nabla u|^{p-2}\nabla u\right)$ is called ν-p-Laplacian. It is the weighted p-Laplace operator for some weight function ν. When $\nu \equiv 1$, $\Delta_p^\nu u = \Delta_p u$ (which is the usual p-Laplace operator) which further reduces to the classical Laplace operator Δ for $p = 2$.

Singular equations present an important role in the development of the mathematical analysis of partial differential equations. As well known this field of boundary value problems arise frequently in the modelling of many real phenomena including, among many others, fluid mechanics, pseudoplastic flow, chemical reactions, population dynamics (Lotka-Volterra systems), combustion, morphogenesis, genetics, etc (see for instance [19]). At the same time, a lot of works are presented to answer the often asked questions on the existence, uniqueness, regularity and the asymptotic behaviour of the solution for the considered problem.

The problem (1.1) does not admit weak solutions and classical methods (see [16]) cannot be applied to prove the existence of a solution. In addition to a necessary compactness argument needed to overcome some difficulties arising from the nonlinearity of the differential operator, and the singularity of the nonlinear terms having variable exponent. To surmount these difficulties we work with the regularisation method and the Schauder's fixed point theorem combined the weighted p-Laplacian technique.

2020 *Mathematics Subject Classification.* 35J92; 35J60; 35D30; 35A02.

Keywords. Weighted p-Laplacian, quasilinear elliptic problem, nonlinear singular terms, existence, weak solution, weighted Sobolev space.

In [14], Lazer and McKenna dealt with the case when f is a continuous function. They proved that the solution belonged to $W_0^{1,2}(\Omega)$ if and only if $\alpha < 3$, while it is not in $C^1(\bar{\Omega})$ if $\alpha > 1$.

In [1], Boccardo and Orsina studied the existence, regularity and nonexistence of solutions to the following problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = f(x) \times u^{-\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $\gamma > 0$ is a real number and f is a nonnegative function belonging to some Lebegue space. The author in [3] studied nonlinear elliptic boundary value problems with singular nonlinearities whose simplest example is

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \frac{f}{u^{\gamma}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N (N \geqslant 2)$, $\gamma > 0, 1 < p < N$, $0 \leqslant f \in L^m(\Omega)$, $m \geqslant 1$, and De Cave proved that there exists a solution u as limit (in a suitable topology) of the sequence $\{u_n\}$. More specifically, the author proved that the above model problem had a suitable solution u for every f in $L^1(\Omega)$ and for every $\gamma > 0$ and how the regularity of u depends on the summability of f, on p and on γ. There are only some important contributions concerning the study of quasilinear elliptic problem with nonlinear singular terms in the weighted Sobolev space involving p-Laplacian operator and rarely do we find work with weighted p-laplacian. One of them is due to Kaushik Bal and Prashanta Garain in [12], they proved some nonexistence results concerning stable solutions to the equation $-\operatorname{div}\left(w(x)|\nabla u|^{p-2}\nabla u\right) = g(x)f(u)$ when $f(u)$ is either $u^{-\delta} + u^{-\gamma}$ with δ, $\gamma > 0$ or $e^{\frac{1}{u}}$ where ω, g are suitables weights functions.

Elliptic problem with nonlinear singular terms is investigated by several authors in the last years (see for example [4, 11, 7, 8, 18]).

In this work, by using the method of regularization and Schauder's fixed point theorem, we prove the existence of weak solution to the nonlinear elliptic problem (1.1).

This article is organized as follows. In part one, we introduce the problem. In part two, we show functional setting and preliminary results. In part three, we state essential assumptions and some fundamental lemmas. In part four, we proved the existence of positive solutions in $W_0^{1,p}(\Omega, \nu)$ for suitable m and $\gamma(x) \geq 1$. The last part is devoted to an application which illustrates our main result.

2. Functional Setting and Preliminary results

Since the problem (1.1) depends on the weight ν, we briefly must recall some various definitions and basic properties of the weighted Lebesgue and Sobolev spaces.

Let ν be a measurable positive and a.e. finite function defined in $\mathbb{R}^N$ and satisfied the following integrability conditions

$$\nu \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \nu^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega), \tag{2.2}$$

$$\nu^{-s} \in L^1_{\text{loc}}(\Omega), \text{ where } \quad s \in \left(\frac{N}{p}, \infty\right) \cap \left[\frac{1}{p-1}, \infty\right). \tag{2.3}$$

For $1 < p < \infty$, we define the weighted Lebesgue space $L^p(\Omega, \nu)$ by

$$L^p(\Omega, \nu) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^p \nu(x) \mathrm{d}x < \infty \right\}.$$

We denote by $L^{p'}(\Omega, \nu^*)$ the conjugate space of $L^p(\Omega, \nu)$, where

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

and where

$$\nu^*(x) = \nu(x)^{1-p'} \quad \text{for all } x \in \Omega.$$

The weighted Sobolev space is defined by

$$W^{1,p}(\Omega,\nu) = \{u \in L^p \text{ and } |\nabla u| \in L^p(\Omega,\nu)\},$$

endowed with the norm

$$\|u\|_{1,p,\nu} = \|u\|_p + \|\nabla u\|_{p,\nu}, \quad \forall u \in W^{1,p}(\Omega,\nu).$$

In the sequel of this paper, the space $W_0^{1,p}(\Omega,\nu)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega,\nu)$. Let s be a real number such that s satisfies the condition (2.3). We define the following critical exponents:

$$p^* = \tfrac{Np}{N-p} \quad \text{for } p < N,$$
$$p_s = \tfrac{ps}{1+s} < p,$$
$$p_s^* = \begin{cases} \frac{ps}{(1+s)N-ps} & \text{if } N > p_s, \\ +\infty & \text{if } N \le p_s. \end{cases}$$

.

Proposition 2.1. *([5]) Let $\Omega \in \mathbb{R}^N$ be an open set of $\mathbb{R}$, and let (2.2) be satisfied, then we have*

$$L^p(\Omega,\nu) \hookrightarrow L^1_{\text{loc}}(\Omega).$$

Proposition 2.2. *([5]) Let condition (2.2) be satisfied, then $\left(W^{1,p}(\Omega,\nu), \|u\|_{1,p,\nu}\right)$ is a separable and reflexive Banach space.*

Proposition 2.3. *([5]) Suppose that conditions (2.2) and (2.3) hold. Then we have the continuous embedding*

$$W^{1,p}(\Omega,\nu) \hookrightarrow W^{1,p_s}(\Omega,\nu).$$

Moreover, we have the compact embedding

$$W^{1,p}(\Omega,\nu) \hookrightarrow\hookrightarrow L^r(\Omega),$$

where $1 \le r < p_s^*$.

Proposition 2.4. *(Hardy-type inequality, see [5]). There exist a weight function $\nu(x)$ on Ω and a parameter q, $1 < q < \infty$, such that the inequality*

$$\left(\int_\Omega \nu(x)|u(x)|^q \, \mathrm{d}x\right)^{1/q} \leqslant C \left(\int_\Omega \nu(x)|\nabla u(x)|^p \, \mathrm{d}x\right)^{1/p},$$

holds for every $u \in W_0^{1,p}(\Omega,\nu)$ with a constant $C > 0$ not depending on u and, moreover, the embedding

$$W_0^{1,p}(\Omega,\nu) \hookrightarrow L^q(\Omega,\nu),$$

expressed by Hardy-type inequality is compact.

Examples of weights.

Power-type weights.

(i): Let Ω be a bounded domain in $\mathbb{R}^N$. Let M be a nonempty closed subset of $\bar{\Omega} = \Omega \cup \partial\Omega$ and denote $d_M(x) = \text{dist}(x, M)$ for $x \in \Omega$ (the distance of the point x from the set M). Let $\rho \in \mathbb{R}$ and let us denote

$$\nu(x) = [d_M(x)]^\rho \quad \text{(power-type weight)} .$$

So, the singularity ($\rho < 0$) or degenerations ($\rho > 0$) can appear on the boundary $\partial\Omega$ of Ω as well as in the interior of the domain. The set M is very often a closed part of the boundary $\partial\Omega$ (i.e., $M \subset \partial\Omega$).

(ii): Let $s = s(t)$ be a continuous positive function defined for $t > 0$ and such that either $\lim_{t\to 0} s(t) = 0$ or $\lim_{t\to 0} s(t) = +\infty$. Let us denote $\nu(x) = s(d_M(x))$. So

(a): If $\lim_{t\to 0} s(t) = 0$ then $W^{k,p}(\Omega) \hookrightarrow W^{k,p}(\Omega, s(d_M))$.

(b): If $\lim_{t\to 0} s(t) = \infty$ then $W^{k,p}(\Omega, s(d_M)) \hookrightarrow W^{k,p}(\Omega)$.

For more information on the properties of spaces with these weights see [21].

A_p weights. The class of A_p weight was introduced by B. Muckenhoupt (see [17]), where he showed that the A_p weights are precisely those weights ν for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^N, \nu)$ to $L^p(\mathbb{R}^N, \nu)$ $(1 < p < \infty)$, that is

$$M : L^p(\mathbb{R}^N, \nu) \to L^p(\mathbb{R}^N, \nu)$$
$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy$$

is bounded if and only if $\nu \in A_p (1 < p < \infty)$, i.e., there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B \nu dx\right)\left(\frac{1}{|B|}\int_B \nu^{-1/(p-1)}dx\right)^{p-1} \leq C,$$

for every ball $B \subset \mathbb{R}^N$. The union of all Muckenhoupt classes A_p is denoted by A_∞, $A_\infty = \bigcup_{p>1} A_p$.

Example of A_p weights.

(i): If $x \in \mathbb{R}^N, \nu(x) = |x|^\alpha$ is in A_p if and only if $-N < \alpha < N(p-1)$ (see Corollary 4.4 in [20]).

(ii): $\nu(x) = e^{\lambda\varphi(x)} \in A_2$, with $\varphi \in W^{1,N}(\Omega)$ and λ is sufficiently small (see Corollary 2.18 in [10]).

For more information about A_p weights, see [9].

We say μ is a bounded Radon measure if $|\mu|(\Omega) = \mu^+(\Omega) + \mu^-(\Omega)$ is bounded.

We define the Weighted p -capacity of a Borel set $E \subset \Omega$ as $\text{cap}(E, \Omega, \nu) = \inf\left\{\int_\Omega (|u(x)|^p + \nu(x)|\nabla u(x)|^p)dx, u \in W_0^{1,p}(\Omega, \nu) : u \geq 1 \text{ a.e. in a neighborh of E}\right\}$. μ is said to be concentrated on a set E of zero ν-p-capacity if $\mu(B) = \mu(E \cap B)$ for every $B \subset \Omega$, with $\text{cap}_{p,\nu}(E) = 0$.

Definition 2.5. A function $u \in W_0^{1,p}(\Omega, \nu)$ is called a weak solution of problem (1.1) if for every $\Omega' \subset \Omega$ there exists a positive constant $C_{\Omega'}$ satisfying $u \geq C_{\Omega'}$ in Ω', one has

$$\int_\Omega u\varphi \mathrm{d}x + \int_\Omega \nu|\nabla u|^{p-2}\nabla u\nabla\varphi \mathrm{d}x = \int_\Omega \frac{h}{u^{\gamma(x)}}\varphi \mathrm{d}x, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{2.4}$$

3. Essential Assumptions and some Fundamental Lemmas

Let us consider the following problem

$$(P^\nu)\begin{cases} L_p^\nu(u) = \frac{h(x)}{u^{\gamma(x)}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

for quasilinear operators in divergence form with the weight ν. Where

$$L_p^\nu(u) := u - \text{div}\, a(x, \nabla u),$$

such that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function $(i.e., a(\cdot, \xi)$ is measurable on Ω, for every $\xi \in \mathbb{R}^N$, and $a(x, \cdot)$ is continuous on $\mathbb{R}^N$, a.e. $x \in \Omega)$ fulfilling the assumptions as follows :

$$a(x, \xi) \cdot \xi \geq \theta\nu(x)|\xi|^p, \tag{3.5}$$

a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where θ is a positive constant.

$$|a(x,\xi)| \leq \delta\nu(x)^{\frac{1}{p}} \left(j(x) + \nu(x)^{\frac{1}{p'}} |\xi|^{p-1}\right), \tag{3.6}$$

a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'}(\Omega)$ and $\delta > 0$.

$$(a(x,\xi) - a(x,\xi')) \cdot (\xi - \xi') > 0, \tag{3.7}$$

a.e. $x \in \Omega$, for every $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$.

$$h \text{ is a nonnegative function belonging to the Lebesgue space } L^m(\Omega) \text{ and } m \geq 1. \tag{3.8}$$

The main difficulty is due to the right hand side $\frac{h(x)}{u^{\gamma(x)}}$, since $u = 0$ on the boundary. In order to overcome this difficulty, similar to the problem discussed in [1]. We consider the following approximating problem is

$$(P_n^\nu) \begin{cases} L_p^\nu(u_n) = \frac{h_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}}, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega, \end{cases}$$

where $h_n(x) = \min\{h(x), n\}$.

Lemma 3.1. *For every $n \in \mathbb{N}^*$ and $h \in L^1(\Omega)$, there exists $u_n \in W_0^{1,p}(\Omega,\nu) \cap L^\infty(\Omega)$ a nonnegative solution to Problem (P_n^ν).*

Proof. This proof follows the technique of [1]. For fixed $n \in \mathbb{N}^*$, we considred a function $\mu \in L^p(\Omega,\nu)$. There exists a unique solution $v \in W_0^{1,p}(\Omega,\nu) \cap L^\infty(\Omega)$ to the following problem

$$\begin{cases} L_p^\nu(v) = \frac{h_n}{\left(|\mu| + \frac{1}{n}\right)^{\gamma(x)}}, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{3.9}$$

To show this we established the associated energy functional $J^\nu : W_0^{1,p}(\Omega,\nu) \to \mathbb{R}$ defined by

$$J^\nu(v) = \int_\Omega v^2 dx + \int_\Omega a(x,\nabla v)\nabla v dx - \int_\Omega \frac{h_n v}{\left(|\mu| + \frac{1}{n}\right)^{\gamma(x)}} dx.$$

It follows easily that J^ν is continuous, strictly convex and coercive so that the existence and the uniqueness of the solution to (3.9) follow by standard arguments. Let us take the mapping $\Gamma : L^p(\Omega,\nu) \to L^p(\Omega,\nu)$ satisfy $\Gamma(\mu) = v$. Let $\varphi = v$ as test function in the weak formulation of (3.9) and using (3.5), thanks to the Sobolev embedding theorem $\left(W^{1,p}(\Omega,\nu) \hookrightarrow L^1(\Omega)\right)$, and the Poincarè inequality, we get that

$$\|v\|_{W^{1,p}(\Omega,\nu)} \leq C n^{\frac{\gamma^+ + 1}{p-1}}.$$

We define $u = \lambda\Gamma u$ with $0 < \lambda \leq 1$, thus $\Gamma u = \frac{u}{\lambda}$. In view of the embedding $W^{1,p}(\Omega,\nu) \hookrightarrow L^p(\Omega,\nu)$ is compact, we get Γ is a compact operator. With a constant C independent of λ, we established

$$\|u\|_{L^p(\Omega,\nu)} \leq \|u\|_{W^{1,p}(\Omega,\nu)} \leq C.$$

By Schauder's fixed point theorem, there exists $u_n \in W_0^{1,p}(\Omega,\nu)$ such that $u_n = \Gamma(u_n)$. So the problem (P_n^ν) admits a solution. Then

$$\frac{h_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \geq 0.$$

By using the the maximum principle in [6], [13], we get that

$$u_n \geq 0, u_n \in L^\infty(\Omega).$$

□

Lemma 3.2. *The sequence $\{u_n\}$ is increasing with respect to $n, u_n > 0$ in Ω, and for every $\Omega' \subset\subset \Omega$, there exists $C_{\Omega'} > 0$ (independent of n) satisfied*

$$u_n(x) \geq C_{\Omega'} > 0 \quad \text{for every } x \in \Omega', \text{ for every } n \in \mathbb{N}^*. \tag{3.10}$$

Proof. This proof follows the technique of [1, 3]. We use $0 \leq h_n \leq h_{n+1}$ and $\gamma(x) > 0$, we find that

$$L_p^\nu(u_n) = \frac{h_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \leq \frac{h_{n+1}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}}.$$

Then

$$L_p^\nu(u_n) - L_p^\nu(u_{n+1}) = h_{n+1} \frac{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)} - \left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)} \left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}}.$$

By taking $\varphi = (u_n - u_{n+1})_+ = \max\{u_n - u_{n+1}, 0\}$ as a test function in the weak formulation of the previous equality and by (3.5), we get

$$0 \leqslant \theta \int_\Omega \nu(x) \left|\nabla (u_n - u_{n+1})_+\right|^p dx \leqslant 0.$$

Thus for every $n \in \mathbb{N}^*$. We get $(u_n - u_{n+1})_+ = 0$ a.e. in Ω, that is, $u_n \leq u_{n+1}$. So the sequence $\{u_n\}$ is increasing with respect to n. For concluding that (3.10) holds for u_1, we use the Lemma 3.1, so $u_1 \in L^\infty(\Omega)$, i.e., there exists a constant $C(\Omega, N)$ such that

$$\|u_1\|_{L^\infty(\Omega)} \leq C \|h_1\|_{L^\infty(\Omega)} \leq C.$$

Then

$$L_p^\nu(u_1) = \frac{h_1}{(u_1 + 1)^{\gamma(x)}} \geq \frac{h_1}{(C+1)^{\gamma(x)}}.$$

Hence $\frac{h_1}{(C+1)^{\gamma(x)}} \geq 0$, $\frac{h_1}{(C+1)^{\gamma(x)}} \neq 0$.

According to the maximum principle in [6] implieds that $u_1 > 0$ in Ω and (3.10) keeps for u_1. Thanks to the monotonicity of u_n, (3.10) keeps for u_n. □

Lemma 3.3. *For $n = 1$ there exists u_1 a solution for problem (P_n^ν) fulfills*

$$\int_\Omega u_1^{-r} \mathrm{d}x < \infty, \quad \forall r < 1. \tag{3.11}$$

Proof. We take $0 < \beta < 1$ fulfilled $u_1 \in C^{1,\beta}(\bar{\Omega})$ and $\|u_1\|_{C^{1,\beta}} \leq C$, $\frac{h_1}{(u_1+1)^{\gamma(x)}} \leq 1$. Then ∇u_1 exists. By the Hopf lemma in [6], we get that $\frac{\partial u_1(x)}{\partial v} > 0$ in $\bar{\Omega}$, here v is the outward unit normal vector of $\partial\Omega$ at x. Finally, due to the proof of lemma in [14], we deduced that

$$\int_\Omega u_1^r \mathrm{d}x < \infty \quad \text{if and only if} \quad r > -1.$$

□

3.1. **Uniqueness Result.** Let u_n and v_n two solutions of (P_n^ν). For every fixed $n \in \mathbb{N}^*$, we have that

$$L_p^\nu(u_n) = \frac{h_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}}.$$

Thus ,

$$L_p^\nu(v_n) = \frac{h_n}{\left(v_n + \frac{1}{n}\right)^{\gamma(x)}}.$$

By subtracting the two previous equality we get

$$L_p^{\nu}(u_n) - L_p^{\nu}(v_n) = h_n \frac{\left(v_n + \frac{1}{n}\right)^{\gamma(x)} - \left(u_n + \frac{1}{n}\right)^{\gamma(x)}}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)} \left(v_n + \frac{1}{n}\right)^{\gamma(x)}}.$$

By choosing $\varphi = (u_n - v_n)_+ = \max\{u_n - v_n, 0\}$ as a test function in the weak formulation of the previous equality and knowing that

$$\left(a(x, \nabla u_n) - a(x, \nabla v_n)\right) \nabla (u_n - v_n)_+ \geq 0,$$

$$\left(\left(v_n + \frac{1}{n}\right)^{\gamma(x)} - \left(u_n + \frac{1}{n}\right)^{\gamma(x)}\right) (u_n - v_n)_+ \leq 0.$$

We obtain that

$$0 \leq \int_{\Omega} \left(a(x, \nabla u_n) - a(x, \nabla v_n)\right) \nabla (u_n - v_n)_+ \, \mathrm{d}x \leq 0.$$

Using (3.5), we have

$$0 \leqslant \theta \int_{\Omega} \nu(x) \left|\nabla (u_n - v_n)_+\right|^p dx \leqslant 0.$$

Then, for every $n \in \mathbb{N}^*$, we find $(u_n - v_n)_+ = 0$ a.e. in Ω, that is,

$$u_n \leq v_n.$$

By changing u_n with v_n and using the same way of the proof , we deduced that

$$v_n \leq u_n.$$

Which implieds that

$$u_n \equiv v_n.$$

Finally, we concluded the uniqueness of solution u_n to problem (P_n^{ν}). Since u_n is increasing in n, we can establish u as the pointwise limit of u_n. There $u \geq u_n$, (3.10) obviously holds for u, too.

4. Main results and Proofs

In this part we state and prove the existence results of this work under assumptions (3.5), (3.6), (3.7), (3.8).

Theorem 4.1. *Let h is a nonnegative function in $L^1(\Omega)$ and $\gamma(x) = 1$, then there exists a solution $u \in W_0^{1,p}(\Omega, \nu)$ for the problem (1.1).*

Proof. Similar to the problem discussed in [1], by changing φ by u_n in the weak formulation of the problem (P_n^{ν}) and using (3.5), we find that

$$\theta \int_{\Omega} \nu(x) \left|\nabla u_n\right|^p dx \leq \int_{\Omega} u_n^2 \mathrm{d}x + \int_{\Omega} a(x, \nabla u_n) \nabla u_n \mathrm{d}x = \int_{\Omega} \frac{h_n u_n}{u_n + \frac{1}{n}} dx,$$

So

$$\int_{\Omega} \nu(x) \left|\nabla u_n\right|^p dx \leq \int_{\Omega} \left|h_n\right| dx \leq \int_{\Omega} |h| dx.$$

Thus

$$\|u_n\|_{W_0^{1,p}(\Omega,\nu)} \leq \|h\|_{L^1(\Omega)}^{\frac{1}{p}}.$$

Hence we get $u \in W^{1,p}(\Omega, \nu)$ and $\vec{V} \in L^{\frac{p}{p-1}}\left((\Omega, \nu), \mathbb{R}^N\right)$ such that

$$\begin{gathered} u_n \text{ converge to } u \text{ weakly in } W^{1,p}(\Omega, \nu) \text{ and strongly in } L^p(\Omega, \nu), \\ u_n \text{ converge to } u \text{ a.e. in } \Omega, \\ a(x, \nabla u_n) \text{ converge to } \vec{V} \text{ weakly in } L^{\frac{p}{p-1}}\left((\Omega, \nu), \mathbb{R}^N\right). \end{gathered} \tag{4.12}$$

And by condition (3.8), we get

$$u_n \text{ converge to } u \quad \text{weakly in } L^1(\Omega).$$

By using (3.9) and for every $\varphi \in C_0^\infty(\Omega)$, we have that

$$0 \leq \left| \frac{h_n \varphi}{u_n + \frac{1}{n}} \right| \leq \frac{\|\varphi\|_{L^\infty}}{C_{\Omega'}} h(x),$$

with $\Omega' = \{x : \varphi \neq 0\}$. Thanks to Lebesgue's dominated convergence theorem, we deduce

$$\lim_{n \to +\infty} \int_\Omega \frac{h_n \varphi}{u_n + \frac{1}{n}} dx = \int_\Omega \frac{h\varphi}{u} dx. \tag{4.13}$$

And so

$$\int_\Omega u_n^2 dx + \int_\Omega a(x, \nabla u_n) \nabla \varphi dx = \int_\Omega \frac{h_n \varphi}{u_n + \frac{1}{n}} dx, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.14}$$

According to (4.12) and (4.14), we find

$$\int_\Omega u^2 dx + \int_\Omega \vec{V} \nabla \varphi dx = \int_\Omega \frac{h\varphi}{u} dx, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.15}$$

As well as , we shall show $\vec{V} = a(x, \nabla u)$ a.e. in Ω. We know that (4.14) and (4.15) satisfied for all $\varphi \in W^{1,p}(\Omega, \nu)$ with compact support. So then in (4.14) we take $\varphi = (u_n - \lambda_2)\lambda_1$ where $\lambda_1 \in C_0^\infty(\Omega), \lambda_1 \geq 0$, and $\lambda_2 \in W^{1,p}(\Omega, \nu)$, we show that

$$\int_\Omega u_n (u_n - \lambda_2) \lambda_1 dx + \int_\Omega a(x, \nabla u_n) \nabla \lambda_1 (u_n - \lambda_2) dx$$
$$+ \int_\Omega \lambda_1 a(x, \nabla u_n) \nabla (u_n - \lambda_2) dx = \int_\Omega \frac{\lambda_1 h_n (u_n - \lambda_2)}{u_n + \frac{1}{n}} dx.$$

It's clear that $\lambda_1 (a(x, \nabla u_n) - a(x, \nabla \lambda_2)) (\nabla u_n - \nabla \lambda_2) \geq 0$, we obtain that

$$\int_\Omega \lambda_1 a(x, \nabla \lambda_2) \nabla (u_n - \lambda_2) dx + \int_\Omega a(x, \nabla u_n) \nabla \lambda_1 (u_n - \lambda_2) dx$$
$$\leq \int_\Omega \lambda_1 a(x, \nabla u_n) \nabla (u_n - \lambda_2) dx + \int_\Omega a(x, \nabla u_n) \nabla \lambda_1 (u_n - \lambda_2) dx$$
$$= \int_\Omega \frac{\lambda_1 h_n (u_n - \lambda_2)}{u_n + \frac{1}{n}} dx. \tag{4.16}$$

We take $n \to \infty$ in (4.16) and by identity (4.15), we get

$$\int_\Omega \lambda_1 a(x, \nabla \lambda_2) \nabla (u - \lambda_2) dx + \int_\Omega \vec{V} \nabla \lambda_1 (u - \lambda_2) dx$$
$$\leq \int_\Omega \frac{\lambda_1 h (u - \lambda_2)}{u} dx = \int_\Omega \vec{V} \nabla (\lambda_1 (u - \lambda_2)) dx$$
$$= \int_\Omega \lambda_1 \vec{V} \nabla (u - \lambda_2) dx + \int_\Omega \vec{V} \nabla \lambda_1 (u - \lambda_2) dx.$$

Which gives that

$$\int_\Omega \lambda_1 \left(a(x, \nabla \lambda_2) - \vec{V} \right) \nabla (u - \lambda_2) dx \leq 0. \tag{4.17}$$

We take $u - \lambda_2 = \varepsilon \psi$ in (4.17), with ψ is an arbitrary function in $W^{1,p}(\Omega, \nu)$ and $\varepsilon > 0$ is a constant, we find that

$$\int_\Omega \lambda_1 \varepsilon \left(a(x, \nabla u - \varepsilon \nabla \psi) - \vec{V} \right) \nabla \psi dx \leq 0,$$

i.e.,

$$\int_\Omega \lambda_1 \left(a(x, \nabla u - \varepsilon \nabla \psi) - \vec{V} \right) \nabla \psi dx \leq 0.$$

By using $\varepsilon \to 0^+$, we have

$$\int_\Omega \lambda_1 \left(a(x,\nabla u) - \vec{V}\right) \nabla\psi dx \leq 0.$$

Thus ψ is an arbitrary function, we have

$$\int_\Omega \lambda_1 \left(a(x,\nabla u) - \vec{V}\right) \nabla\psi dx = 0.$$

Noting $\psi = px$, with $p \in \mathbb{R}^N$, and then

$$\int_\Omega \lambda_1 p \left(a(x,\nabla u) - \vec{V}\right) dx = 0.$$

Which yields that $\vec{V} = a(x,\nabla u)$ a.e. in Ω. As a result u is a weak solution of problem (1.1) with $h(x) \in L^1(\Omega)$. □

Theorem 4.2. *Let μ is a nonnegative Radon measure concentrated on a Borel set E of zero ν-p-capacity, and that g_n is a bounded sequence of nonnegative $L^1(\Omega)$ functions which converges to μ in the narrow topology of measures. Assume that u_n is a solution of problem (P_n^ν) with the non-homogeneous function $h_n = g_n(x)$. Then*

$$\lim_{n\to\infty} \int_\Omega |u_n|^p \, dx = 0.$$

Proof. By using Theorem 4.1, we obtained the solution u_n of problem (P_n^ν) with $h_n = g_n$ is bounded in $W_0^{1,p}(\Omega,\nu)$. From Lemma 5.1 in [2] and since the set E has zero ν-p-capacity, then there exists a function $\Psi_\sigma \in C_0^\infty(\Omega)$ for any real number $\sigma > 0$, satisfying

$$0 \leq \Psi_\sigma \leq 1, \quad 0 \leq \int_\Omega (1-\Psi_\sigma)\, d\mu \leq \sigma, \quad \int_\Omega |\nabla\Psi_\sigma|^p \, dx \leq \sigma. \tag{4.18}$$

Taking g_n converges to μ in the narrow topology of measure, from (4.18) we find that

$$\lim_{n\to\infty} \int_\Omega g_n (1-\Psi_\sigma)\, dx = \int_\Omega (1-\Psi_\sigma)\, d\mu \leq \sigma. \tag{4.19}$$

We note that $T_1(u_n) = \min\{u_n, 1\}$. Changing φ by $T_1(u_n)(1-\Psi_\sigma)$ in the weak formulation of (P_n^ν) with a non-homogeneous function g_n, we get

$$\int_\Omega u_n T_1(u_n)(1-\Psi_\sigma)\, dx +$$

$$\int_\Omega a(x,\nabla u_n)\nabla T_1(u_n)(1-\Psi_\sigma)\, dx - \int_\Omega a(x,\nabla u) T_1(u_n)\nabla\Psi_\sigma dx$$

$$= \int_\Omega \frac{g_n T_1(u_n)(1-\Psi_\sigma)}{u_n + \frac{1}{n}} dx$$

$$\leq \int_\Omega \frac{g_n (1-\Psi_\sigma) u_n}{u_n} dx = \int_\Omega g_n (1-\Psi_\sigma)\, dx. \tag{4.20}$$

As $\|u_n\|_{W_0^{1,p}(\Omega,\nu)} \leq C$, we assume that u_n is any subsequence fulfilled $u_n \to u$ in $W^{1,p}(\Omega,\nu)$ and $u_n \to u$ in $L^p(\Omega,\nu)$. We show that the two limits in the theorem hold for any such sub-sequence. We obtain that

$$\left|\int_\Omega T_1(u_n)\, a(x,\nabla u_n)\nabla\Psi_\sigma dx\right| \leq \left|\int_\Omega a(x,\nabla u_n)\nabla\Psi_\sigma dx\right|$$

$$\leq \delta \nu(x)^{\frac{1}{p}} \left(j(x) + \|\nabla u_n\|^{p-1}_{L^p(\Omega,\nu)}\right) \|\nabla\Psi_\sigma\|_{L^p(\Omega,\nu)}$$

$$\leq C \|\nabla\Psi_\sigma\|_{L^p(\Omega,\nu)}. \tag{4.21}$$

Thanks to weak lower semi-continuity and by (4.18)-(4.21), we get that

$$\int_\Omega a(x,\nabla u)\nabla T_1(u)\,(1-\Psi_\sigma)\,dx \le \sigma + \sigma^{\frac{1}{p}}. \tag{4.22}$$

Letting $\sigma \to 0^+$, we get that

$$0 \le \int_\Omega a(x,\nabla u)\nabla T_1(u)\,dx \le 0.$$

Which yields $u = 0$ a.e. in Ω. Lastly, the proof of the theorem is proved. □

Theorem 4.3. *Let h is a nonnegative function in $L^m(\Omega)(h \neq 0)(m > 1)$ and that $1 < \gamma(x) \le \gamma^+ < 2 - \frac{1}{m}$. Then there exists a solution $u \in W_0^{1,p}(\Omega,\nu)$ for the problem (P^ν) such that*

$$\int_\Omega u\varphi dx + \int_\Omega a(x,\nabla u)\nabla\varphi dx = \int_\Omega \frac{h\varphi}{u^{\gamma(x)}}dx, \quad \forall\varphi \in C_0^\infty(\Omega).$$

Proof. Changing φ by u_n in the weak formulation of (P_n^ν), according to Holder's inequality and Lemma 3.3 and (3.5), we get

$$\theta\int_\Omega \nu(x)\,|\nabla u_n|^p\,dx \le \int_\Omega u_n^2 dx + \int_\Omega a(x,\nabla u_n)\nabla u_n dx = \int_\Omega \frac{h_n u_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}dx$$

$$\int_\Omega \nu(x)\,|\nabla u_n|^p\,dx \le \int_\Omega \frac{h_n}{u_n^{\gamma^+-1}}dx \le \int_\Omega \frac{h}{u_1^{\gamma^+-1}}dx$$

$$\le \|h\|_{L^m}\left\|u_1^{1-\gamma^+}\right\|_{L^{m'}} \le c\|h\|_{L^m}. \tag{4.23}$$

Since $(1-\gamma^+)m' = (1-\gamma^+)\frac{m}{m-1} > -1$ and the condition $1 < \gamma^+ < 2 - \frac{1}{m}$, we have

$$\|u_n\|_{W_0^{1,p}(\Omega,\nu)} \le c\|h\|_{L^m(\Omega)}^{\frac{1}{p}}.$$

By using (4.23), we concluded that there exist $u \in W_0^{1,p}(\Omega,\nu)$ and $\vec{V} \in L^{\frac{p}{p-1}}\left((\Omega,\nu),\mathbb{R}^N\right)$ such that

$$\begin{gathered} u_n \text{ converge to } u \text{ weakly in } W^{1,p}(\Omega,\nu) \text{ and strongly in } L^p(\Omega,\nu), \\ u_n \text{ converge to } u \text{ a.e. in } \Omega, \\ a(x,\nabla u_n) \text{ converge to } \vec{V} \text{ weakly in } L^{\frac{p}{p-1}}\left((\Omega,\nu),\mathbb{R}^N\right). \end{gathered} \tag{4.24}$$

From condition (3.8), we get

$$u_n \text{ converge to } u \text{ weakly in } L^1(\Omega).$$

For every $\varphi \in C_0^\infty(\Omega)$. From Lemma 3.2, we obtain that

$$0 \le \left|\frac{h_n\varphi}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}\right| \le \frac{\|\varphi\|_{L^\infty}}{C'_\Omega}h(x).$$

So due to Lebesgue's dominated convergence theorem, we get that

$$\lim_{n\to+\infty}\int_\Omega \frac{h_n\varphi}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}dx = \int_\Omega \frac{h\varphi}{u^{\gamma(x)}}dx. \tag{4.25}$$

Since u_n satisfies :

$$\int_\Omega u_n\varphi dx + \int_\Omega a(x,\nabla u_n)\nabla\varphi dx = \int_\Omega \frac{h_n\varphi}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}dx, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{4.26}$$

Taking $n \to \infty$ in (4.26) and by using (4.24) and (4.25) we get

$$\int_\Omega u\varphi dx + \int_\Omega \vec{V}\nabla\varphi dx = \int_\Omega \frac{h\varphi}{u^{\gamma(x)}}dx, \quad \forall\varphi \in C_0^\infty(\Omega).$$

Using the same method in the proof of Theorem 4.1, we show that the problem (1.1) has a weak solution in $W_0^{1,p}(\Omega,\nu)$. □

5. Application

Let $\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\right\}$ and let the weight functions

$$\nu(x,y) = \left(x^2+y^2\right)^{-1/2}$$

is A_4-weight, $p=4$, and the functions $a : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$a((x,y),\xi) = h_1(x,y)|\xi|^2\xi,$$

where $h_1(x,y) = 4e^{\left(x^2+y^2\right)}$, and $h(x,y) = 1/2\left(x^2+y^2\right)^{1/4}$. Let the operator

$$L_4^{\nu}(u(x,y)) := u - \operatorname{div} a((x,y),\nabla u(x,y)).$$

Therefore, by Theorem 4.1, the problem

$$(P^{\nu}) \begin{cases} L_4^{\nu}(u(x,y)) = \frac{h(x,y)}{u(x,y)}, & in\,\Omega, \\ u(x,y) = 0, & on\,\partial\Omega, \end{cases}$$

has a solution $u \in W_0^{1,4}(\Omega,\nu)$.

Conflict of interest The authors declared that they have no conflict of interest.

Data Availability Not applicable.

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Received 14/08/2025; Revised 24/06/2026