Methods of Functional Analysis and Topology
Vol. 32 (2026), no. 3, pp. 266–272
doi.org/10.31392/MFAT-npu26_3.2026.07

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

OPTIMAL DOMAIN OF THE VOLTERRA TYPE OPERATORS ACTING IN BERGMAN SPACES

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Abstract. The concept of optimal domains for certain operators serves as a powerful tool for refining inequalities and extending the action of these operators beyond their classical domains. G. Curbera and W. Ricker first introduced and studied the optimal domain of the classical Cesro operator acting on Hardy spaces. Later, C. Bellavita and collaborators investigated the optimal domain of Volterra-type operators T_g mapping into Hardy spaces. Motivated by these works, we consider the optimal domain for which the Volterra-type operator T_g maps into the Bergman space A^p, and we explore its structural properties in detail.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For any $g \in H(\mathbb{D})$, define two integral operators S_g and T_g on $H(\mathbb{D})$ by

$$S_g f(z) \triangleq \int_0^z f'(w)g(w)\,dw, \quad \text{and} \quad T_g f(z) \triangleq \int_0^z f(w)g'(w)\,dw.$$

Let M_g be the multiplication operator given by $M_g(f)(z) = f(z)g(z)$. Then the following decomposition holds:

$$M_g(f)(z) = f(0)g(0) + T_g(f)(z) + S_g(f)(z).$$

For a function $f \in H(\mathbb{D})$ with power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, the Cesáro operator $\mathcal{C} : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by

$$\mathcal{C}(f)(z) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n.$$

The Volterra-type operator T_g generalizes the classical Cesàro operator. In fact, for the specific choice $g(z) = \log\left(\frac{1}{1-z}\right)$, we have the identity $T_g f(z) = z\mathcal{C}(f)(z)$.

The operator T_g was first studied on Hardy spaces by Pommerenke [16]. Since then, its boundedness, compactness, and spectral properties on various analytic function spaces have attracted considerable attention. In particular, Aleman and Cima [3], as well as Aleman and Siskakis [1, 2], studied T_g on Hardy and Bergman spaces. Later, Siskakis and Zhao [18] considered its action on the space BMOA, while Xiao [19] investigated both S_g and T_g on the Q_p spaces. For a broader treatment and further references on these integral operators, we refer the reader to [9, 10, 11, 12, 13, 14, 15, 20, 21].

2020 *Mathematics Subject Classification.* 47B38, 30H, 30H20.
Keywords. Optimal domain, Volterra operator, Integration operator, Bergman spaces.
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This work was supported by NNSF of China (Grant Nos. 11801094, 62072119) and NNSF of Guangdong Province (Grant No. 2023A1515010001).

Denote the normalized area measure on $\mathbb{D}$ by $dA(z) = \frac{1}{\pi}\,dxdy$. For $p \geq 1$, the Bergman space A^p consists of all analytic functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p\, dA(z)\right)^{1/p} < \infty.$$

Let $0 < \alpha < \infty$. The α-Bloch space $\mathcal{B}^\alpha$ is defined as the set of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty.$$

If

$$\lim_{|z|\to 1^-} (1-|z|^2)^\alpha |f'(z)| = 0,$$

we say that $f \in \mathcal{B}_0^\alpha$. In particular, the Bloch space and the little Bloch space correspond to $\mathcal{B} = \mathcal{B}^1$ and $\mathcal{B}_0 = \mathcal{B}_0^1$, respectively.

For more information on $\mathcal{B}^\alpha$, we refer the reader to [23] and [22], as well as the references therein.

Let H^∞ denote the space of all bounded analytic functions on $\mathbb{D}$, that is,

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

In accordance with the notation used in [7], we define the optimal domain of the generalized Volterra operator T_g acting on the space A^p as

$$[T_g, A^p] := \{f \in H(\mathbb{D}) : T_g(f) \in A^p\}.$$

It is clear that $[T_g, A^p]$ is a linear subspace of $H(\mathbb{D})$. If g is constant, then trivially $[T_g, A^p] = H(\mathbb{D})$. For non-constant g, we endow this space with the norm

$$\|f\|_{[T_g,A^p]} := \|T_g(f)\|_{A^p}.$$

In the sections that follow, we investigate the basic structural properties of this normed space.

Clearly, if g is constant, then $[T_g, A^p] = H(\mathbb{D})$, and these result is trivial. In this paper, we will always assume that g is not constant.

The structure of this paper is organized as follows. In section 2, we investigate the structural properties of the optimal domain. In particular, we establish that the point evaluations are uniformly bounded on compact subsets, which leads to the proof of Theorem 2.2. Finally, Section 3 focuses on the action of the multiplication operator on the optimal domain and examines the density of polynomials in $[T_g, A^p]$.

For convenience, we write $A \sim B$ to mean that there exist constants $c, d > 0$ such that $cB \leq A \leq dB$. Moreover, $p \in \mathcal{X} \setminus \mathcal{Y}$ indicates that p belongs to $\mathcal{X}$ but not to $\mathcal{Y}$.

2. The Structure of the Optimal Domain

Our main goal in this section is to show that the space $[T_g, A^p]$ is a Banach space.

Let $1 \leq p < \infty$. It is straightforward to verify that $([T_g, A^p], \|\cdot\|_{[T_g,A^p]})$ is a normed linear space. Indeed, if $\|f\|_{[T_g,A^p]} = 0$, then $\|T_g f\|_{A^p} = 0$, which implies $T_g f \equiv 0$. Since $g' \not\equiv 0$, it follows that $f \equiv 0$.

Moreover, for any $\beta \in \mathbb{C}$, we have $\|\beta f\|_{[T_g,A^p]} = |\beta|\|f\|_{[T_g,A^p]}$. To verify the triangle inequality, we apply Minkowski's inequality:

$$\begin{aligned}
\|f+h\|_{[T_g,A^p]} &= \left(\int_{\mathbb{D}} |(T_g f + T_g h)(z)|^p\, dA(z)\right)^{1/p} \\
&\leq \left(\int_{\mathbb{D}} |T_g f(z)|^p\, dA(z)\right)^{1/p} + \left(\int_{\mathbb{D}} |T_g h(z)|^p\, dA(z)\right)^{1/p} \\
&= \|f\|_{[T_g,A^p]} + \|h\|_{[T_g,A^p]}.
\end{aligned}$$

We now establish the boundedness of point evaluations.

Lemma 2.1. *Let $1 \le p < \infty$. Then the point evaluations on $[T_g, A^p]$ are uniformly bounded on every compact subset $K \subset \mathbb{D}$.*

Proof. Let $z_0 \in \mathbb{D}$ and $h \in A^p$. By [23, p. 74] and the Cauchy integral formula, we have

$$|h'(z_0)| \le \frac{1}{(1-|z_0|^2)^{1+\frac{2}{p}}} \|h\|_{A^p}. \tag{2.1}$$

Now let $f \in [T_g, A^p]$, so $T_g f \in A^p$, and

$$|f(z_0)\cdot g'(z_0)| = |(T_g f)'(z_0)| \le \frac{1}{(1-|z_0|^2)^{1+\frac{2}{p}}} \|T_g f\|_{A^p} = \frac{1}{(1-|z_0|^2)^{1+\frac{2}{p}}} \|f\|_{[T_g,A^p]}.$$

Thus, the boundedness of point evaluations reduces to whether we can divide the above inequality by $|g'(z_0)|$.

Let K be a compact subset of $\mathbb{D}$. If $z_0 \in K$ and $g'(z_0) \neq 0$, then

$$|f(z_0)| \le \frac{1}{|g'(z_0)|(1-|z_0|)^{1+\frac{2}{p}}} \|f\|_{[T_g,A^p]}.$$

Since g' is continuous, the denominator is bounded away from zero on K, so the evaluation functional is bounded in a neighborhood of z_0.

If $g'(z_0) = 0$, then there exists an annulus A_{z_0} centered at z_0 such that

$$\min_{\zeta \in A_{z_0}} |g'(\zeta)| > \varepsilon > 0.$$

Since $|f|$ is subharmonic, we apply the sub-mean value property. Using $dA(w)$ for the normalized area measure and $|A_{z_0}|$ for the area of A_{z_0}, we get

$$\begin{aligned} |f(z_0)| &\le \frac{1}{|A_{z_0}|} \int_{A_{z_0}} |f(w)|\, dA(w) \\ &\le \frac{1}{|A_{z_0}|} \int_{A_{z_0}} \frac{1}{(1-|w|)^{1+\frac{2}{p}} \min_{\zeta\in A_{z_0}} |g'(\zeta)|} \|f\|_{[T_g,A^p]}\, dA(w) \\ &= \frac{\|f\|_{[T_g,A^p]}}{\varepsilon |A_{z_0}|} \int_{A_{z_0}} \frac{1}{(1-|w|)^{1+\frac{2}{p}}}\, dA(w) = C\|f\|_{[T_g,A^p]}. \end{aligned}$$

Here, the constant $C > 0$ depends only on z_0. Since f is continuous, the point evaluations are uniformly bounded in a neighborhood of z_0. As K is compact, it can be covered by finitely many such neighborhoods. Therefore, the point evaluations are uniformly bounded on K. □

Theorem 2.2. *Let $1 \le p < \infty$ and $g \in \mathcal{B}$. Then the optimal domain $[T_g, A^p]$, equipped with the norm $\|\cdot\|_{[T_g,A^p]}$, is a Banach space.*

Proof of Theorem 2.2. We examine the completeness of $\big([T_g, A^p], \|\cdot\|_{[T_g,A^p]}\big)$. Let $\{f_n\}$ be a Cauchy sequence in $[T_g, A^p]$, i.e.,

$$\|f_n - f_m\|_{[T_g,A^p]} = \|T_g(f_n - f_m)\|_{A^p} \to 0, \tag{2.2}$$

as $m, n \to \infty$. By Lemma 2.1, we have

$$|(f_n - f_m)(z)| \lesssim \|f_n - f_m\|_{[T_g,A^p]},$$

so $\{f_n\}$ is uniformly Cauchy on every compact subset of $\mathbb{D}$. Hence, there exists a function $f \in H(\mathbb{D})$ such that $f_n \to f$ locally uniformly on $\mathbb{D}$.

Since A^p is a Banach space and $\{f_n\}$ is Cauchy in $[T_g, A^p]$, there exists a function $k \in A^p$ such that

$$\|T_g f_n - k\|_{A^p} \to 0 \quad \text{as } n \to \infty.$$

Because point evaluations of derivatives in Bergman spaces are uniformly bounded on compact sets, for every $z \in \mathbb{D}$, we have

$$|T_g'(f_n)(z) - k'(z)| \leq \frac{\|T_g f_n - k\|_{A^p}}{(1-|z|^2)^{1+\frac{2}{p}}} \to 0.$$

Thus, $T_g'(f_n)(z) \to k'(z)$ for each $z \in \mathbb{D}$. On the other hand,

$$T_g'(f_n)(z) = f_n(z)g'(z) \to f(z)g'(z) \quad \text{as } n \to \infty.$$

Hence, $k'(z) = f(z)g'(z)$, and integrating both sides yields

$$k(z) = \int_0^z f(w)g'(w)\,dw + C.$$

Since $T_g f_n(0) = 0$ and $k(0) = 0$, we must have $C = 0$, and so $k = T_g(f)$, implying that $f \in [T_g, A^p]$. Therefore, the space is complete. □

The optimal domain $[T_g, A^p]$ is generally much larger than A^p. For any $g \in \mathcal{B}$ and $f \in A^p$, since

$$\|T_g f\|_{A^p} \lesssim \|g\|_{\mathcal{B}} \|f\|_{A^p},$$

it follows that $A^p \subseteq [T_g, A^p]$. A natural question arises: under what conditions does this inclusion become an equality, and when is it strict? These issues are addressed in Propositions 2.4 and 2.6. In addition, we provide a characterization of the boundedness of the multiplication operator on $[T_g, A^p]$, as well as a description of the density property (see Proposition 3.3).

Next, we consider the inclusion between $[T_g, A^p]$ and A^p. To analyze this, the bounded-below property of T_g plays a crucial role.

Before proceeding, we recall the definition: an operator T is said to be bounded below from A^p to A^p if there exists a constant $C > 0$ such that

$$\|Tf\|_{A^p} \geq C\|f\|_{A^p}.$$

Lemma 2.3. [5] *Let $1 \leq p < \infty$. Then T_g is bounded below on A^p if and only if there exist constants $c > 0$ and $\delta > 0$ such that*

$$\left|\{z \in \mathbb{D} : |g'(z)|(1-|z|^2) > c\} \cap S(I)\right| > \delta |I|^2, \tag{2.3}$$

for all arcs $I \subset \partial\mathbb{D}$, where $S(I)$ denotes the Carleson square over I.

Proposition 2.4. *Let $1 \leq p < \infty$ and $g \in \mathcal{B}$ satisfy (2.3). Then $A^p = [T_g, A^p]$. Moreover, if $g \in \mathcal{B}$ does not satisfy (2.3), then the inclusion is strict: $A^p \subsetneqq [T_g, A^p]$.*

Proof. By Lemma 2.3, if g satisfies (2.3), then T_g is bounded below on A^p, hence $[T_g, A^p] \subseteq A^p$.

On the other hand, since $g \in \mathcal{B}$, a standard result (see [2]) asserts that the Volterra-type operator T_g is bounded on A^p, and satisfies

$$\|T_g f\|_{A^p} \lesssim \|g\|_{\mathcal{B}} \|f\|_{A^p}, \quad \forall f \in A^p.$$

Thus $A^p \subseteq [T_g, A^p]$, and it follows that $A^p = [T_g, A^p]$. Ramey and Ullrich constructed such functions g (see [17]).

If $g \notin \mathcal{B}$ or does not satisfy (2.3), then T_g is not bounded below, and the inclusion is strict. □

Remark 2.5. In particular, if $g \in \mathcal{B}_0$, then T_g is compact [2], which implies that it is not bounded below. Therefore, the inclusion is strict: $A^p \subsetneqq [T_g, A^p]$.

Bellavita et al. [6] showed that for $1 \le p < \infty$,

$$H^p = \bigcap_{g \in \mathrm{BMOA}} [T_g, H^p].$$

Although the Hardy space H^p may be regarded as a limiting case of the weighted Bergman space A^p_α as $\alpha \to -1$, many fine analytic properties of A^p_α do not carry over to H^p. Thus, the analogy is limited.

In the Bergman space A^p, our result differs from the Hardy space case: whether or not g satisfies (2.3) plays a key role. If g does satisfy (2.3), then

$$A^p = \bigcap_{g \in \mathcal{B}} [T_g, A^p].$$

For the opposite situation, we have the following:

Proposition 2.6. *If $g \in \mathcal{B}$ does not satisfy (2.3), then*

$$A^p \subsetneqq \bigcap_{g \in \mathcal{B}} [T_g, A^p].$$

Proof. Suppose $g \in \mathcal{B}$ does not satisfy (2.3), and assume by contradiction that

$$A^p = \bigcap_{g \in \mathcal{B}} [T_g, A^p].$$

Then for any $f \in \bigcap_{g \in \mathcal{B}} [T_g, A^p]$, we must have $f \in A^p$, which would imply

$$\|f\|_{A^p} \lesssim \|T_g f\|_{A^p}.$$

But this inequality holds if and only if g satisfies (2.3), leading to a contradiction. Therefore,

$$A^p \subsetneqq \bigcap_{g \in \mathcal{B}} [T_g, A^p].$$

□

3. On Pointwise Multipliers and Density

In Theorem 2.2, we showed that the space $\big([T_g, A^p], \|\cdot\|_{[T_g, A^p]}\big)$ is a Banach space. Since the study of multiplication operators provides insight into the structure of analytic function spaces and plays a crucial role in operator theory, we now consider the boundedness of multiplication operators acting on $[T_g, A^p]$.

Theorem 3.1. *Let $1 \le p < \infty$ and $g \in \mathcal{B}$. The multiplication operator*

$$M_h : [T_g, A^p] \to [T_g, A^p]$$

is bounded if and only if $h \in H^\infty$. Moreover, there exists a constant $C > 0$ such that

$$C\|h\|_\infty \le \|M_h\| \le \|h\|_\infty.$$

Proof. Let $f \in [T_g, A^p]$ and $h \in H^\infty$. Observe that

$$T_g(hf)(z) = \int_0^z h(\zeta) f(\zeta) g'(\zeta)\, d\zeta = S_h(T_g f)(z),$$

where S_h denotes the multiplication operator induced by h acting on A^p. By [4, Proposition 2.4], if $h \in H^\infty$, then $\|S_h(f)\|_{A^p} \le \|h\|_\infty \|f\|_{A^p}$. Therefore,

$$\begin{aligned} \|M_h(f)\|_{[T_g, A^p]} &= \|T_g(hf)\|_{A^p} = \|S_h(T_g f)\|_{A^p} \\ &\le \|h\|_\infty \|T_g(f)\|_{A^p} = \|h\|_\infty \|f\|_{[T_g, A^p]}. \end{aligned}$$

Thus, M_h is bounded on $[T_g, A^p]$.

Conversely, suppose $M_h : [T_g, A^p] \to [T_g, A^p]$ is bounded. For any $z \in \mathbb{D}$, let δ_z denote the point evaluation functional. By Lemma 2.1, δ_z is a bounded linear functional on $[T_g, A^p]$, and since the constant function belongs to $[T_g, A^p]$, we have $\|\delta_z\| \neq 0$. Then, for any $f \in [T_g, A^p]$ with $\|f\|_{[T_g,A^p]} = 1$, we obtain

$$|h(z)f(z)| = |\delta_z(M_h(f))| \leq \|\delta_z\|\|M_h\|.$$

Taking the supremum over all such f, we deduce

$$|h(z)| \leq \|M_h\| \quad \text{for all } z \in \mathbb{D},$$

which implies $h \in H^\infty$. □

Proposition 3.2. *Let $1 \leq p < \infty$ and $g \in H^\infty$. Then $A^p = [M_g, A^p]$.*

Proof. If $g \in H^\infty$, then $\|M_g f\|_{A^p} \lesssim \|g\|_{H^\infty}\|f\|_{A^p}$, so $A^p \subseteq [M_g, A^p]$. Moreover, by [5, Theorem 3.9], the operator M_g is bounded below on A^p, which implies $[M_g, A^p] \subseteq A^p$. Hence, equality holds. □

Proposition 3.3. *Let $g \in \mathcal{B}$ and $p \geq 1$. Then the polynomials $\mathbb{P}[z]$ are dense in $[T_g, A^p]$ if and only if A^p is dense in $[T_g, A^p]$.*

Proof. We observe that $\overline{\mathbb{P}[z]} = [T_g, A^p]$ if and only if $\overline{A^p} = [T_g, A^p]$. Suppose $\overline{\mathbb{P}[z]} = [T_g, A^p]$. Since $\mathbb{P}[z] \subseteq A^p$, it follows that $[T_g, A^p] \subseteq \overline{A^p}$. As A^p is complete, we obtain $\overline{A^p} = A^p$. Therefore:

$$\overline{A^p} \subseteq \overline{[T_g, A^p]} \subseteq \overline{\overline{A^p}} = A^p \subseteq [T_g, A^p] \subseteq \overline{A^p}.$$

Conversely, suppose A^p is dense in $[T_g, A^p]$. For any $\varepsilon > 0$ and $f \in [T_g, A^p]$, there exists $h \in A^p$ such that $\|f - h\|_{[T_g,A^p]} < \varepsilon/2$. Since $\mathbb{P}[z]$ is dense in A^p (see [8, Proposition 1.3]), there exists a polynomial $p(z)$ such that $\|h - p\|_{A^p} < \varepsilon/(2C)$, where C is the norm of T_g on A^p. Then,

$$\begin{aligned}\|f - p\|_{[T_g,A^p]} &\leq \|f - h\|_{[T_g,A^p]} + \|h - p\|_{[T_g,A^p]} \\ &= \|f - h\|_{[T_g,A^p]} + \|T_g(h - p)\|_{A^p} \\ &\leq \frac{\varepsilon}{2} + C\|h - p\|_{A^p} \lesssim \varepsilon.\end{aligned}$$

Hence, $\mathbb{P}[z]$ is dense in $[T_g, A^p]$. □

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Received 30/08/2025; Revised 19/10/2025