Methods of Functional Analysis and Topology
Vol. 32 (2026), no. 3, pp. 273–280
doi.org/10.31392/MFAT-npu26_3.2026.08

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

EXISTENCE AND REGULARITY RESULTS FOR d-AND $\overline{\partial}$-COMPLEXES ON DOMAINS WITH REAL ANALYTIC BOUNDARIES

SALOMON SAMBOU AND SHABAN KHIDR

Abstract. Existence and regularity theorems for d and $\overline{\partial}$-problems are established on unbounded domains with real analytic boundaries. These theorems are then combined to prove analogous existence and regularity results for the $\partial\overline{\partial}$-equation on half-space in $\mathbb{C}^{n+1}$. Global vanishing theorems for de Rham and Dolbeault cohomologies on manifolds separated by real-analytic hypersurfaces are respectively obtained. In the proofs, the use of Mayer-Vietoris sequences and the covering of unbounded domains by exhaustive sequences of balls are among the key techniques.

1. Introduction

Let $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{R}$. We say that f is real-analytic at a boundary point $x_0 \in \partial\Omega$ if f has a convergent power series expansion in a neighborhood of x_0 in $\mathbb{C}^n$. A set $S \subset \mathbb{C}^n$ is called a real-analytic hypersurface if for each $p \in S$ there are a neighborhood U of p and a real analytic defining function $\varphi : U \to \mathbb{R}$ such that

(i) $U \cap S = \{x \in U : \varphi(x) = 0\}$,
(ii) $d\varphi(x) \neq 0$ on U.

An open set $\Omega \subset \mathbb{C}^n$ is said to have a real-analytic boundary $\partial\Omega$ if for every point $x_0 \in \partial\Omega$ there exists an open neighborhood U such that $S = U \cap \partial\Omega$ is a real-analytic hypersurface so that $U \cap \partial\Omega$ lies on one side of S. Without loss of generality, we can choose U and φ so that

$$U \cap \Omega = \{x \in U : \varphi(x) > 0\}.$$

This means that Ω looks locally like the upper half-space in $\mathbb{C}^n$, and the boundary is smooth (in fact, real analytic). So we can always flatten the boundary locally using real analytic coordinate changes. Examples of such domains:

Example 1.1. If $D = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im} z_2 > 0\}$. Then $\partial D = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im} z_2 = 0\}$ is real-analytic.

Example 1.2. Exterior domains: $\Omega = \mathbb{R}^{n+1} \setminus \overline{D}$, where D is a bounded open domain with analytic boundary.

Example 1.3. Special half-spaces: $\Omega = \{(x', x_{n+1}) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \mid x_{n+1} > \varphi(x')\}$, where φ is a real-analytic function on $\mathbb{R}^n$.

The complex analytic properties of bounded pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$ can differ very much from those of strictly pseudoconvex domains (see [7]). For bounded pseudoconvex domains with real-analytic boundaries, however, the complex analysis seems to be in many respects more similar to the strictly pseudoconvex case. This is indicated by the following results.

2020 *Mathematics Subject Classification.* 32W05, 32C35, 58A12, 32U40.
Keywords. Real-analytic hypersurfaces, Existence theorems, Contractible manifolds, $\overline{\partial}$-operator, de Rham complex, Extensible currents.

Theorem 1.4 ([8]). *Let D be a relatively compact pseudoconvex domain with smooth real-analytic boundary in $\mathbb{C}^n$. Then one has:*

(1) *Subelliptic estimates hold on D for the $\overline{\partial}$-Neumann problem for (p,q)-forms.*
(2) *$\overline{D}$ has a Stein neighborhood basis.*

An obvious obstruction to subelliptic estimates is the presence of a germ of an analytic variety in the boundary of D. Indeed, examples show that local regularity of the $\overline{\partial}$-Neumann problem fails when there are complex varieties in the boundary; see Catlin [5] and Diedrich-Pflug [9].

If ∂D is real-analytic near a point $x_0 \in \partial D$, then the absence of germs of q-dimensional complex analytic varieties in the boundary near x_0 is necessary and sufficient condition for the existence of subelliptic estimates on $(0,q)$-forms; Kohn [12]. This was first proved by combining a sufficient condition from Kohn's theory of ideals of subelliptic multipliers with a theorem of Diederich-Fornaess [8].

On the other hand, there exists a pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$-boundary which does not have a Stein neighborhood basis.

Recall that a linear partial differential operator P, with $\mathcal{C}^\infty$-coefficients, defined on a smooth compact manifold M without boundary, is said to be hypoelliptic if for any open set $V \subset M$ and any distribution u on V such that $Pu \in \mathcal{C}^\infty(V)$, necessarily $u \in \mathcal{C}^\infty(V)$. The following example shows that the $\overline{\partial}$-hypoellipticity breaks down when analytic varieties are present in the boundary.

Example 1.5. Define

$$D := \Big\{(z_1, z_2) \in \mathbb{C}^2; |z_1| < 1, \mathrm{Re} z_2 + \mathcal{O}(z_2^2) < 0\Big\}.$$

D is a pseudoconvex domain with $\mathcal{C}^\infty$-boundary in $\mathbb{C}^2$ and hence $\Delta := \{(z_1, 0) : |z_1| \leq 1\} \subset D$. Then $\overline{\partial}$ is not hypoelliptic for $(0,1)$-forms on D. This result was generalized to (p,q)-forms on pseudoconvex domains in $\mathbb{C}^n$, see Catlin [5] or Diedrich-Pflug [9].

If D is strongly pseudoconvex or pseudoconvex with real analytic boundary, then $\overline{\partial}$ is hypoelliptic on D; see [11].

Souhaibou et al [15] solved the d and $\overline{\partial}$-equations on half-spaces for $\mathcal{C}^\infty$-forms having boundary values in the sense of currents. Consequently, they showed that the equation $\partial\overline{\partial} u = f$ is solvable whenever the datum f is of class $\mathcal{C}^\infty$ and has a boundary value in the sense of currents. Thereby, L^p-Sobolev regularity results for $\partial\overline{\partial}$ have been recently studied by Khidr and Sambou in [10]. We refer the interested readers to the introduction of [10] for more closely related results.

The main purpose of this paper is to prove existence and regularity theorems for the exterior differential operator d (resp. the Cauchy-Riemann operator $\overline{\partial}$) acting on smooth forms on unbounded domains in $\mathbb{R}^{n+1}$ (resp. $\mathbb{C}^{n+1}$). The regularity results are still valid for forms with boundary values in the sense of currents. As a result, we obtain the same regularity for the solution of the $\partial\overline{\partial}$-equation on unbounded domains having real-analytic boundaries for datum of class $\mathcal{C}^\infty$.

The de Rham vanishing theorem for real-analytic hypersurfaces generally refers to the vanishing of certain de Rham cohomology groups of their complements. More precisely, if $S \subset \mathbb{R}^{n+1}$ is a real-analytic hypersurface that divides $\mathbb{R}^{n+1}$ into two connected components $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$ and the de Rham cohomology $H^j(S) = 0$, then we have

$$H^j(\mathbb{R}^{n+1}_+) = H^j(\mathbb{R}^{n+1}_-) = 0 \quad \text{for } 1 \leq j \leq n+1.$$

In complex settings, similar results hold for the $\overline{\partial}_b$-cohomology groups under real-analytic assumptions. Some consequences will also be obtained. These particular results are moreover generalized to contractible (resp. Stein) manifolds.

2. Function spaces

Let X be a differentiable manifold of dimension n and $M \subset X$ be an open subset of X. We can define on M several spaces of functions, differential forms, and currents; (cf. [10]).

- $\mathcal{C}^\infty(M)$ is the space of $\mathcal{C}^\infty$-smooth functions on M with its classical Fréchet topology.
- $\mathcal{C}^\infty(\overline{M})$ is the subspace of $\mathcal{C}^\infty$-smooth functions up to the boundary of M; this is the space of the restrictions to M of functions in $\mathcal{C}^\infty(X)$. We endow $\mathcal{C}^\infty(\overline{M})$ with the Fréchet topology induced by $\mathcal{C}^\infty(X)$.
- $\mathcal{D}(M)$ is the space of smooth compactly supported functions on M, which is a topological vector space with the standard inductive limit topology.
- $\mathcal{E}^k(M)$ is the Fréchet space of k-forms of class $\mathcal{C}^\infty$ on M, where $k \geq 0$ is a finite integer. The associated de Rham cohomology group is denoted as usual by $H^k(M)$.
- $\mathcal{D}^k(M)$ is the space of forms in $\mathcal{E}^k(M)$ with compact supports in M.
- $\mathcal{D}'^k(M)$ is the space of k-currents on M, the topological dual of the space $\mathcal{D}^{n-k}(M)$, endowed with the topology of uniform convergence on bounded subsets of $\mathcal{D}^{n-k}(M)$. In particular, a distribution is a 0-current.
- $\check{\mathcal{D}}'^k(M)$ is the space of all extensible k-currents T on M. Such currents T are defined as restrictions to M of currents $\tilde{T}$ on X. The associated de Rham cohomology group is denoted by $\check{H}^k(M)$.

For complex setting, let $M \subset X$ be an open subset of an n-dimensional complex manifold X. We will use the following spaces of (p,q)-forms and currents associated to the previous function spaces. For fixed $0 \leq p \leq n$ and $1 \leq q \leq n$, we denote by:

- $\mathcal{E}^{p,q}(M)$ the Fréchet space of (p,q)-forms of class $\mathcal{C}^\infty$ on M endowed with the topology of uniform convergence of the forms and all their derivatives on compact subsets of M. For every $k \in \{0,1,\ldots,2n\}$, we have

$$\mathcal{E}^k(M) = \bigoplus_{p+q=k} \mathcal{E}^{p,q}(M).$$

The complex structure of M splits the exterior differential operator

$$d : \mathcal{E}^k(M) \to \mathcal{E}^{k+1}(M)$$

uniquely into

$$d = \partial + \overline{\partial}$$

and the $\partial\overline{\partial}$-operator is then defined as

$$\partial\overline{\partial} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p+1,q+1}(M).$$

- $\mathcal{D}^{p,q}(M)$ the space of (p,q)-forms of class $\mathcal{C}^\infty$ and compactly supports in M.
- $\mathcal{D}'^{p,q}(M)$ the space of currents of bidegree (p,q) on M. $\mathcal{D}'^{p,q}(X)$ is, by definition, the topological dual space to the space $\mathcal{D}^{n-p,n-q}(X)$ with the $\mathcal{C}^\infty$-topology.
- $\check{\mathcal{D}}'^{p,q}(M)$ the space of extensible currents of bidegree (p,q) on M.

The associated $\overline{\partial}$-cohomology groups are denoted respectively by $H^{p,q}(M)$, $H_c^{p,q}(M)$, $H_{\text{cur}}^{p,q}(M)$, $\check{H}^{p,q}(M)$, and $H_{W^s}^{p,q}(M)$. Finally, we denote by $\tilde{H}^q(\cdot)$ (resp. $\tilde{H}^{p,q}(\cdot)$) the q-th (resp. (p,q)-th) cohomology group for $\mathcal{C}^\infty$ forms with boundary values in the currents sense.

If $\mathcal{S}$ is a real hypersurface in X, both d and $\overline{\partial}$-operators induce in a natural way complexes of differential operators on S. The boundary cohomology groups associated to these tangential complexes are denoted respectively by $H^j(\mathcal{S})$ and $H^{p,j}(\mathcal{S})$ (cf. [1], [2]).

3. Existence theory

Theorem 3.1. *Let $\mathcal{S}$ be a real-analytic hypersurface that devides $\mathbb{R}^{n+1}$ into two connected components $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$. Let $B = B(O,r)$ be a ball centered at the origin of $\mathbb{R}^{n+1}$ with radius $r > 0$. Assume that r is big enough such that $B_1 = B \cap \mathbb{R}^{n+1}_+ \neq \emptyset$ and $B_2 = B \cap \mathbb{R}^{n+1}_- \neq \emptyset$. Then,*

$$H^q(B_1) = H^q(B_2) = 0 \text{ for } 1 \leq q \leq n+1 \tag{3.1}$$

and

$$\tilde{H}^q(B_1) = \tilde{H}^q(B_2) = 0 \text{ for } 1 \leq q \leq n+1. \tag{3.2}$$

Proof. Let f be a q-closed form of class $\mathcal{C}^\infty$ on B_1 (resp. B_2). B_1 (resp. B_2) can be covered by a finite number of convex open sets $(U_j)_{j=1,\dots,p}$ or diffeomorphic domains to the half space. On each U_j, we can find a $\mathcal{C}^\infty$ $(q-1)$-form ϕ_j such that

$$f = d\phi_j \quad \text{in } U_j.$$

Let $(\gamma_j)_{1,\dots,p}$ be a partition of unity subordinate to the covering U_j, then we can set

$$u = \sum_{j=1}^{p} \gamma_j \phi_j.$$

Hence

$$du = \sum_{j=1}^{p} d\gamma_j \wedge \phi_j + f \text{ and } \theta = \sum_{j=1}^{p} d\gamma_j \wedge \phi_j$$

has compact support in B_1 (resp. B_2) included in B. Indeed,

$$\operatorname{supp}(d\gamma_j \wedge \phi_j) \subset \operatorname{supp}(\gamma_j) \Subset U_j \subset B_1 (\text{ resp. } B_2).$$

Since θ has compact support, we can extend it by zero on $\mathbb{R}^{n+1} \setminus B_1$ (resp. $\mathbb{R}^{n+1} \setminus B_2$) which provides a d-closed form still denoted by θ. Therefore,

$$\theta = d\eta,$$

where η is a $(q-1)$-form of class $\mathcal{C}^\infty$ with compact support in $\mathbb{R}^{n+1}$. Finally, we have

$$f = du - d\eta \mid_{B_1} = d(u - \eta \mid_{B_1}) = d\psi \quad \text{in } B_1 (\text{ resp. } B_2).$$

This proves (3.1). The proof of (3.2) follows similarly thanks to [15, Theorem 3.2]. □

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded domain with real-analytic boundary $\partial\Omega$. Let $\eta \in \mathcal{E}^q(\Omega)$ with $d\eta = 0$. Then,*

(i) *There exists a smooth $(q-1)$-form $\omega \in \mathcal{E}^{q-1}(\Omega)$ such that*

$$d\omega = \eta.$$

(ii) *If η admits a boundary value in the sense of currents, then so does the solution ω.*

Proof. Take a sequence $(B(O,n))_{n \geq n_0}$ of balls, work as in the case of half space (cf. [10]), and solve in $\mathbb{R}^{n+1}_+$ (or in $\mathbb{R}^{n+1}_-$), depending on the side of Ω from S. Indeed, set $\Omega_k = B_k \cap \Omega$ and apply Theorem 3.1 (1), we have a solution u_k to the equation

$$du_k = \eta \mid_{\Omega_k} \quad \text{in } \Omega_k.$$

It is clear that

$$d(u_{k+2} - u_{k+1}) = 0 \quad \text{in } \Omega_{k+1},$$

then,

$$\exists v_k \text{ such that } u_{k+2} - u_{k+1} = dv_k \text{ in } \Omega_{k+1}.$$

Now, let χ_k be a $\mathcal{C}^\infty$ function in Ω with compact support in Ω_{k+2} and $\chi_k \equiv 1$ in Ω_k. Hence,

$$u_{k+2} - d\chi_k v_k = u_{k+1} - (1-\chi_k)v_k \text{ on } \Omega_{k+1}.$$

Then, we set

$$\begin{aligned}\omega_{k+2} &= u_{k+2} - d\chi_k v_k,\\ \omega_{k+1} &= u_{k+1} - (1-\chi_k)v_k\\ &= u_{k+1} \text{ on } \Omega_k.\end{aligned}$$

$$\Rightarrow \omega = \lim_{k\to+\infty} \omega_k$$

is a $\mathcal{C}^\infty$ solution to $d\omega = \eta$ in Ω. This proves (i). Assertion (ii) follows by repeating the same arguments and using (3.2) instead of (3.1). □

As a particular application of Theorem 3.2, we have:

Corollary 3.3. *Let $\mathcal{S}$ be a real-analytic and unbounded hypersurface of $\mathbb{R}^{n+1}$. Assume that $\mathcal{S}$ divides $\mathbb{R}^{n+1}$ into two connected components $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$. Then,*

$$H^j(\mathcal{S}) = 0,\ 1 \le j \le n+1.$$

Proof. We have the sequence

$$\begin{aligned}0 \to H^0(\mathbb{R}^{n+1}) \to H^0(\overline{\mathbb{R}^{n+1}_+}) \oplus H^0(\overline{\mathbb{R}^{n+1}_-}) \to H^0(\mathcal{S}) \to H^1(\mathbb{R}^{n+1}) \to\\ \to \cdots \to H^n(\mathcal{S}) \to H^{n+1}(\mathbb{R}^{n+1}) \to H^{n+1}(\overline{\mathbb{R}^{n+1}_+}) \oplus H^{n+1}(\overline{\mathbb{R}^{n+1}_-}) \to 0\end{aligned}$$

According to Theorem 3.2, for $1 \le j \le n+1$, we have

$$H^j(\overline{\mathbb{R}^{n+1}_+}) = H^j(\mathbb{R}^{n+1}_+) = 0,$$

$$H^j(\overline{\mathbb{R}^{n+1}_-}) = H^j(\mathbb{R}^{n+1}_-) = 0,$$

and

$$H^j(\mathbb{R}^{n+1}) = 0 \ \text{ for } j \ge 1.$$

Then

$$H^j(\mathcal{S}) = 0 \ \text{ for } j \ge 1.$$

□

We have also the analogue complex case of these results.

Theorem 3.4. *Let $\Omega \subset \mathbb{C}^{n+1}$ be an unbounded pseudoconvex domain with real-analytic boundary. Then, there exists a solution $\omega \in \mathcal{E}^{p,j-1}(\overline{\Omega})$ to $\overline{\partial}\omega = \eta$ for every $\eta \in \mathcal{E}^{p,j}(\overline{\Omega})$ with $\overline{\partial}\eta = 0$. Moreover, if η admits a boundary value, in the sense of currents, then so does the solution ω.*

Corollary 3.5. *Let $\mathcal{S}$ be a real-analytic, unbounded, hypersurface of $\mathbb{C}^{n+1}$. Assume that $\mathcal{S}$ divides $\mathbb{C}^{n+1}$ into two connected components $\mathbb{C}^{n+1}_+$ and $\mathbb{C}^{n+1}_-$. Then*

$$H^{p,j}_{\overline{\partial}_b}(\mathcal{S}) = 0,\ 1 \le j \le n+1.$$

Combining results of Theorems 3.2 and 3.4, we obtain the following corollary.

Corollary 3.6. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a contractible unbounded pseudoconvex domain with real-analytic boundary. Then, for every d-closed form $f \in \mathcal{E}^{p,j}(\Omega)$ there exists a form $u \in \mathcal{E}^{p,j-1}(\Omega)$ such that $\partial\overline{\partial}u = f$ in Ω. Assume further that $H^j(b\Omega) = 0$ for $1 \le j \le 2n$. Then, if f has a boundary value, in the sense of currents, then so does u.*

4. Vanishing theorems for real-analytic hypersurfaces complements

In this section, we prove some vanishing theorems for de Rham (resp. Dolbeault) cohomology groups on contractible (resp. Stein) manifolds including real-analytic, unbounded, hypersurfaces. The proof relies on the Mayer-Vietoris sequence and topological properties of contractible spaces and submanifolds.

Theorem 4.1. *Let M be a contractible manifold of dimension n and $\mathcal{S}$ be a real-analytic hypersurface. Assume that $\mathcal{S}$ divides M into two connected components M^+ and M^-. If $H^j(\mathcal{S}) = 0$ for $1 \leq j \leq n-1$, then*

$$H^j(M^+) = H^j(M^-) = 0 \text{ for } 1 \leq j \leq n. \tag{4.3}$$

Proof. By [2, Theorem 3.1], we have the long exact Mayer-Vietoris sequence

$$\begin{aligned} 0 \to H^0(M) \to H^0(\overline{M^+}) \oplus H^0(\overline{M^-}) \to H^0(S) \\ \to H^1(M) \to H^1(\overline{M^+}) \oplus H^1(\overline{M^-}) \to H^1(S) \cdots \end{aligned}$$

Since M is contractible, then $H^j(M) = 0$ for all $j > 0$ (cf. [6, Theorem 8.3.8]). Therefore, for $1 \leq j \leq n-1$, the above sequence reduces to

$$0 \to H^j(\overline{M^+}) \oplus H^j(\overline{M^-}) \to H^j(S) \to 0.$$

Thus, for $1 \leq j \leq n-1$, we have

$$H^j(\overline{M^+}) \oplus H^j(\overline{M^-}) \cong H^j(S) = 0,$$

and hence

$$H^j(\overline{M^+}) = H^j(\overline{M^-}) = 0 \quad \text{for } 1 \leq j \leq n-1.$$

Since $H^j(M^\pm) = H^j(\overline{M^\pm})$, then $H^j(M^\pm) = 0$ for $1 \leq j \leq n-1$. To handle the case $j = n$, consider the portion of the Mayer-Vietoris sequence:

$$H^{n-1}(S) \to H^n(M) \to H^n(\overline{M^+}) \oplus H^n(\overline{M^-}) \to H^n(S).$$

We are given $H^{n-1}(S) = 0$. Since S is an $(n-1)$-dimensional manifold, we also have $H^n(S) = 0$. In addition, $H^n(M) = 0$. Therefore,

$$0 \to H^n(\overline{M^+}) \oplus H^n(\overline{M^-}) \to 0,$$

which implies

$$H^n(M^+) = H^n(M^-) = 0.$$

Combining both results, we obtain (4.3). □

Corollary 4.2. *In the conditions of the above theorem, we have:*

$$\check{H}^j(M^+) = \check{H}^j(M^-) = 0 \quad \text{for } 1 \leq j \leq n-1.$$

Proof. By Theorems 3.1 and 12.1 in [2], we have the double sequence:

$$\begin{array}{ccccccccc} 0 & \to & H^0(M) & \to & H^0(\overline{M^+}) \oplus H^0(\overline{M^-}) & \to & H^0(S) \cdots & \to & H^{n-1}(S) \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^0_{\text{cur}}(M) & \to & \check{H}^0(M^+) \oplus \check{H}^0(M^-) & \to & H^0_{\text{cur}}(S) \cdots & \to & H^{n-1}_{\text{cur}}(S) \cdots \end{array}$$

with a commutative diagram. All vertical arrows are isomorphisms by de Rham's theorem for smooth manifolds, so $H^j_{\text{cur}}(S) \simeq H^j(S) = 0$ and $H^j_{\text{cur}}(M) = 0$ for $j \geq 1$. Thus the second sequences reduces to

$$0 \to \check{H}^j(M^+) \oplus \check{H}^j(M^-) \to 0 \ \forall\ j \geq 1,$$

which implies that

$$\check{H}^j(M^+) = \check{H}^j(M^-) = 0 \quad \forall\ j \geq 1.$$

□

We turn now to the complex setting.

Theorem 4.3. *Let M be a Stein manifold of complex dimension n and $\mathcal{S}$ be a real analytic hypersurface of M. Assume that $\mathcal{S}$ divides M into two connected components M^+ and M^-. If $H^{p,j}(\mathcal{S}) = 0$, then for $j \geq 1$, we have*

$$H^{p,j}(\overline{M^+}) = H^{p,j}(\overline{M^-}) = 0, \tag{4.4}$$

$$\check{H}^{p,j}(M^+) = \check{H}^{p,j}(M^-) = 0. \tag{4.5}$$

Proof. Following [1, Theorem 1], we have the exact sequence

$$\begin{aligned} 0 \to H^{p,0}(M) \to H^{p,0}(\overline{M^+}) \oplus H^{p,0}(\overline{M^-}) \to H^{p,0}(S) \\ \to H^{p,1}(M) \to H^{p,1}(\overline{M^+}) \oplus H^{p,1}(\overline{M^-}) \to H^{p,1}(S) \cdots \end{aligned} \tag{4.6}$$

Since M is Stein, we have $H^{p,j}(M) = 0$. Then (4.4) follows exactly as the proof of Theorem 4.1. To prove (4.5), we combine (4.6) with [13, Propositions 9 and 10] to obtain the double sequence:

$$\begin{array}{ccccccccc} 0 & \to & H^{p,0}(M) & \to & H^{p,0}(\overline{M^+}) \oplus H^{p,0}(\overline{M^-}) & \to & H^{p,0}(S) \cdots & \to & H^{n-1,0}(S) \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^{p,0}_{\text{cur}}(M) & \to & \check{H}^{p,0}(M^+) \oplus \check{H}^{p,0}(M^-) & \to & H^{p,0}_{\text{cur}}(S) \cdots & \to & H^{n-1,0}_{\text{cur}}(S) \cdots \end{array}$$

where the vertical arrows are isomorphisms; see [14, Theorems 3.1, 3.2, and 3.4]. Thus $H^{p,j}_{\text{cur}}(M) \simeq H^{p,j}(M) = 0$; because M is Stein, and $H^{p,j}_{\text{cur}}(S) \simeq H^{p,j}(S) = 0$; because S is real-analytic. Therefore, the below sequence becomes

$$0 \to \check{H}^{p,j}(M^+) \oplus \check{H}^{p,j}(M^-) \to 0, \quad j \geq 1,$$

this implies (4.5). □

A direct consequence of Theorems 4.1 and 4.3, respectively, is the following corollary.

Corollary 4.4. *For all $j \geq 1$, we have*

$$\tilde{H}^j(\mathbb{R}^{n+1}_+) = \tilde{H}^j(\mathbb{R}^{n+1}_-) = 0,$$

and

$$\tilde{H}^{p,j}(\mathbb{C}^{n+1}_+) = \tilde{H}^{p,j}(\mathbb{C}^{n+1}_-) = 0.$$

Proof. We present the proof in the real case; the complex case follows by the same reasoning, replacing the de Rham complex with the Dolbeault complex. Let f be a representative of a class in $\tilde{H}^j(\mathbb{R}^{n+1}_+)$. That is, f is a d-closed smooth j-form on $\mathbb{R}^{n+1}_+$ with boundary values in the sense of currents. Then f defines a current on $\overline{\mathbb{R}^{n+1}_+}$, and by extension, an extensible current on all of $\mathbb{R}^{n+1}$ supported in $\overline{\mathbb{R}^{n+1}_+}$. Apply Corollary 4.2 for $M = \mathbb{R}^{n+1}$, we see that $\check{H}^j(\mathbb{R}^{n+1}_+) = 0$ for $j \geq 1$. Therefore, there exists an extensible current T on $\mathbb{R}^{n+1}$ such that

$$dT = f \quad \text{in } \mathbb{R}^{n+1}_+.$$

Now, regularize T via convolution with an approximate identity supported in the interior of $\mathbb{R}^{n+1}_+$ to obtain smooth forms $T_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^{n+1}_+)$ with boundary values (since convolution preserves the boundary behavior in this sense). Then

$$dT_\varepsilon \to f \quad \text{in the sense of currents.}$$

But since f is smooth and $dT_\varepsilon \to f$, for small enough ε, we have $dT_\varepsilon = f$ identically (by standard regularization arguments and support conditions).

Thus, $f = dT_\varepsilon$ with T_ε smooth and having boundary values in the current sense. Therefore, the cohomology class of f in $\tilde{H}^j(\mathbb{R}^{n+1}_+)$ is zero. The same method is applied for the lower half-space $\mathbb{R}^{n+1}_-$. Hence, $\tilde{H}^j(\mathbb{R}^{n+1}_+) = \tilde{H}^j(\mathbb{R}^{n+1}_-) = 0$ for $j \geq 1$. □

Let Θ be a paracompactifying family of supports on the manifold M. Then there is a paracompactifying family of supports on S, induced by intersection with S, which we denote by $\Theta(S)$, and corresponding induced families on M^+ and M^-, which we continue to denote by Θ; see e.g., [2]. The corresponding de Rham (resp. Dolbeault) cohomology group of smooth j (resp. (p,j))-forms with supports in Θ is denoted as usual by $H^j_\Theta(\cdot)$ (resp. $H^{p,j}_\Theta(\cdot)$). Then Theorems 4.1, 4.3, and their Corollaries in which every $H^j(\cdot)$ (resp. $H^{p,j}(\cdot)$) is replaced by $H^j_\Theta(\cdot)$ (resp. $H^{p,j}_\Theta(\cdot)$) are still hold; according to Theorem 2 in [3] (or Proposition 2 in [4]) and Proposition 4 in [4].

References

[1] A. Andreotti and D. C. Hill, *E.E. Levi convexity and the Hans Lewy problem I*, Ann. Scuola Norm. Sup. Pisa **26** (1972), 325–363.

[2] A. Andreotti, D. C. Hill, S. Lojasiewicz and B. Mackichan, *Complexes of partial differential operators: The Mayer–Vietoris sequence*, Invent. Math. **26** (1976), 43–86, doi:10.1007/BF01390133.

[3] A. Andreotti and M. Nacinovich, *Noncharacteristic hypersurfaces for complexes of differential operators*, Ann. Mat. Pura Appl. **125** (1980), 18–83, doi:10.1007/BF01789406.

[4] A. Andreotti, G. Fredricks and M. Nacinovich, *On the absence of Poincaré lemma in tangential Cauchy–Riemann complexes*, Ann. Scuola Norm. Sup. Pisa, Classe di Scienze, 4$^{\text{e}}$ série, **8** (1981), no. 3, 365–404.

[5] D. Catlin, *Necessary conditions for subellipticity and hypoellipticity for the $\overline{\partial}$-Neumann problem on pseudoconvex domains*, Recent developments in several complex variables, Ann. Math. Stud., vol. 100, Princeton Univ. Press, 1981, pp. 93–100, doi:10.1515/9781400881543-007.

[6] L. Conlon, *Differentiable Manifolds*, 2nd ed., Birkhäuser Boston, Cambridge, MA, 2001, doi:10.1007/978-0-8176-4767-4.

[7] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: An example with nontrivial Nebenhülle*, Math. Ann. **225** (1977), 275–292, doi:10.1007/BF01425243.

[8] K. Diederich and J. E. Fornæss, *Pseudoconvex domains with real-analytic boundary*, Ann. Math. **107** (1978), 371–384, doi:10.2307/1971120.

[9] K. Diederich and P. Pflug, *Necessary conditions for hypoellipticity for the $\overline{\partial}$-problem*, In: *Recent Developments in Several Complex Variables*, Ann. Math. Stud., Princeton Univ. Press, **100** (1981), 151–154, doi:10.1515/9781400881543-011.

[10] S. Khidr and S. Sambou, *L^p-theory for the $\partial\overline{\partial}$-equation and isomorphism results*, Electronic Research Archive **33** (2025), 68–86, doi:10.3934/era.2025004.

[11] J. J. Kohn, *Global regularity for $\overline{\partial}$ on weakly pseudoconvex manifolds*, Trans. Amer. Math. Soc. **181** (1973), 273–292, doi:10.1090/S0002-9947-1973-0344703-4.

[12] J. J. Kohn, *Subellipticity of the $\overline{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions*, Acta Math. **142** (1979), 79–122, doi:10.1007/BF02395058.

[13] M. Nacinovich and G. Valli, *Tangential Cauchy–Riemann complexes on distributions*, Ann. Mat. Pura Appl. **146** (1996), 123–160, doi:10.1007/BF01762362.

[14] S. Sambou, *Équation de Cauchy–Riemann pour les courants prolongeables: applications*, C. R. Acad. Sci. Paris, Sér. I **332** (2001), 497–500, doi:10.1016/S0764-4442(01)01871-7.

[15] S. Sambou, W. O. Ingoba and S. Sambou, *Resolution of $\overline{\partial}$ for differential forms with boundary values in the sense of currents defined on a half space of $\mathbb{C}^n$*, Nonlinear analysis, geometry and applications, Trends in Mathematics, Birkhäuser, Cham, 2022, pp. 399–406, doi:10.1007/978-3-031-04616-2_16.

Salomon Sambou: ssambou@univ-zig.sn

Departement of Mathematics, UFR of Sciences and Technologies, University Assane Seck of Ziguinchor, BP: 523, Senegal

Shaban Khidr: skhidr@uj.edu.sa & skhidr@science.bsu.edu.eg

Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia

Department of Mathematics and Computer Sciences, Faculty of Science, Beni-Suef University, 62511 Beni-Suef, Egypt

Received 25/08/2025; Revised 16/12/2025