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MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

$f_\lambda-$STATISTICAL CONVERGENCE FOR DOUBLE SEQUENCES IN TOPOLOGICAL GROUPS

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Abstract. In this paper, we define and study new notions of $f_\lambda-$statistical convergence and $f_\lambda-$statistical Cauchy sequence for double sequences in topological groups. Furthermore, we gave the relationships between these concepts with inclusion theorems. Additionally, we considered these relationships for two different modulus functions.

1. Introduction

In the 1935 first edition of Zygmund's monograph, statistical convergence was called "almost convergence" [54]. Later, the concept of statistical convergence was introduced by Steinhaus [51] and Fast [20] for real and complex sequences. Schoenberg [50] showed the connection between the concept of statistical convergence and the summability method. The concept of statistical convergence is based on the concept of natural density. These concepts have been studied by many mathematicians (see Buck [7], Freedman and Sember [21]). Fridy [22] established an equivalent criterion for statistically convergent real sequences, analogous to the Cauchy criterion of convergence. The notion of statistical convergence was defined by Maddox [31] for sequences in any Hausdorff locally convex topological vector space. Salat [44] defined the concept of statistical boundedness of real number sequences. He also analysed the basic properties of these sequences. On the other hand, the concept of statistical convergence in Banach spaces was studied by Kolk [26]. Later on, statistical convergence in Banach spaces was studied by many researchers (see Bhardwaj and Bala [5], Connor *et al.* [10]). Recently, statistical convergence in probabilistic normed spaces has been studied by Rahaman and Mursaleen [42].

Statistical convergence emerged as a dynamic area of study in summability theory. Researchers from a variety of disciplines, including Fourier analysis, number theory, measure theory, ergodic theory, and turnpike theory have examined statistical convergence.

The concept of summability in topological groups was initially studied by Prullage ([38], [39], [40], [41]). This concept was later studied by Çakalli and Thorpe ([11], [12]). Later, the concept of statistical convergence in topological groups was first defined by Çakalli ([13], [14]). Recently, the topological properties of statistical convergence have been studied by Di Maio and Kočinac [19].

The generalized de la Vallee-Poussin mean was defined by Leindler [27]. Mursaleen [34] introduced the concepts of $\lambda-$density and $\lambda-$statistical convergence. Recently, many authors have studied different properties of this convergence (see Aziz and Ghosal [3], Bhardwaj and Dhawan [4], Borgohain and Savas [6], Çolak [16]). Later, the concept of $\lambda-$statistical convergence in topological groups was studied by many researchers (see Hazarika and Esi [25], Savas [48]).

In 1953, Nakano introduced the idea of the modulus function [35]. This concept was later used to generated some sequence spaces by Maddox [30], Mölder [32] and Ruckle [43]. The relationship between statistically convergent and strongly summable was then provided by Connor [9] using the modulus function. Later, Aizpuru *et al.* [1] $f-$density

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and $f-$statistical convergence were defined and examined. Furthermore, Çolak [17] used two unbounded modulus functions to define statistical convergence. García-Pacheco [23] worked on topological module characteristics. Statistical convergence in topological modules was later characterized by García-Pacheco and Kama [24]. Additionally, by taking a more comprehensive view of the modulus function, Sarikaya and Altin [45] investigated statistical convergence in topological groups. The statistical convergence of double sequences in topological groups was later investigated by Çakalli and Savas [15]. $I_\lambda-$statistical convergence of double sequences in topological groups was introduced by Savas [49].

In this work, we used the modulus function in topological groups to define the concepts $f_\lambda-$statistical convergence and $f_\lambda-$statistical Cauchy for double sequences. Furthermore, we gave the relationships between these concepts with inclusion theorems. Additionally, we considered these relationships for two different modulus functions.

2. Definitions and Preliminaries

This section contains the main definitions and concepts that we used in our study. Moreover, Z will be treated additively and as an abelian topological Hausdorff group in this study. Let Z also fulfill the first countability axiom.

Definition 2.1. The density of $\mathcal{A}$ is defined as follows:

$$\delta(\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathcal{A}}(k).$$

In this case, $\chi_{\mathcal{A}}$ stands for characteristic function for $\mathcal{A}$. If $\mathcal{A} \subset \mathbb{N}$ is a finite set and $\delta(\mathbb{N} \setminus \mathcal{A}) = 1 - \delta(\mathcal{A})$, then it is evident that $\delta(\mathbb{N}) = 1$ and $\delta(\mathcal{A}) = 0$, (see [7], [51]).

Definition 2.2. A real number sequence $z = (z_k)$ is said to be statistically convergent to $\mathcal{L}$, if

$$\delta\left(\{k \in \mathbb{N} : |z_k - \mathcal{L}| \geq \varepsilon\}\right) = 0$$

for any $\varepsilon > 0$. Here, $S - \lim z_k = \mathcal{L}$ is written, and S is used to represent the set of all statistically convergent sequences, (see [22], [29], [44]).

Definition 2.3. A function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ that satisfies the following requirements is called a modulus function:

i) $f(z) = 0 \Leftrightarrow z = 0$,
ii) $f(z_1 + z_2) \leq f(z_1) + f(z_2)$ for every $z_1, z_2 \in \mathbb{R}^+$,
iii) f is increasing,
iv) f is continuous from the right at 0, (see [35], [43]).

All of these functions are obviously continuous. A modulus may be unbounded or bounded. When $0 < p \leq 1$, modulus functions include, for instance, $f_1(z) = \frac{z}{z+1}$, $f_2(z) = z^p$ and $f_3(z) = \log(z+1)$. Here, f_3 is an unbounded modulus function, whereas f_1 and f_2 are bounded modulus functions.

Definition 2.4. For each $\varepsilon > 0$, a double sequence of real numbers $z = (z_{k,l})$ is said to be convergent in Pringsheim's sense or $P-$convergent if there exists $M \in \mathbb{N}$ such that $|z_{k,l} - \mathcal{L}| < \varepsilon$, $k, l \geq M$. Here, $\mathcal{L}$ is known as the Pringsheim limit, and it is represented by $P - \lim z = \mathcal{L}$, (see [37]).

Definition 2.5. Let $\mathcal{A} \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $\mathcal{A}_{m,n}$ be the numbers (i, j) in $\mathcal{A}$ such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of $\mathcal{A}$ is described as

$$P-\liminf_{m,n\to\infty}\frac{\mathcal{A}_{m,n}}{mn}=\delta^2(\mathcal{A}).$$

In this case where the sequence $\left(\frac{\mathcal{A}_{m,n}}{mn}\right)_{m,n=1,1}^{\infty,\infty}$ has the limit, we say that $\mathcal{A}$ has a natural density and is defined as follows:

$$P-\lim_{m,n\to\infty}\frac{\mathcal{A}_{m,n}}{mn}=\delta^2(\mathcal{A}).$$

So, let $\mathcal{A}=\left\{\left(i^2,j^2\right):i,j\in\mathbb{N}\right\}$, where $\mathbb{N}$ is the set of natural numbers. This means that the natural density of the set $\mathcal{A}$ is zero or the

$$\delta^2(\mathcal{A})=P-\lim_{m,n\to\infty}\frac{\mathcal{A}_{m,n}}{mn}\leq P-\lim_{m,n\to\infty}\frac{\sqrt{m}\sqrt{n}}{mn}=0.$$

The sequence $z=(z_{k,l})$ is said to be statistically convergent if there is a complex number $\mathcal{L}$ such that for every $\varepsilon>0$

$$\lim_{m,n\to\infty}\frac{1}{mn}\left|\{(k,l):k\leq m,l\leq n:|z_{k,l}-\mathcal{L}|\geq\varepsilon\}\right|=0$$

in which case, we say that $z=(z_{k,l})$ is statistically convergent to $\mathcal{L}$ (see [33]).

If we take $n=m$ in the double density definition defined by Mursaleen and Edely [33], we obtain the density concept for two-dimensional sets from the density concept defined for $k-$dimensional sets given by Christoper [8]. More recent developments on double sequences can be found in (Çolak [18], León-Saavedra *et al.* [28], Tripathy and Sarma [52], Zhu *et al.* [53]).

Definition 2.6. As m and n get closer to ∞, let $\lambda=(\lambda_m)$ and $\mu=(\mu_n)$ be two nondecreasing sequences of positive real integers that both tend to ∞. Additionally, assume that $\lambda_{m+1}\leq\lambda_m+1$, $\lambda_1=0$ and $\mu_{n+1}\leq\mu_n+1$, $\mu_1=0$. Δ stands for the collection of these sequences (λ,μ).

Let $\mathcal{A}\subseteq\mathbb{N}\times\mathbb{N}$. The number

$$\delta_\lambda^2(\mathcal{A})=\lim_{m,n\to\infty}\frac{1}{\lambda_{m,n}}\left|\{k\in I_m,l\in I_n:(k,l)\in\mathcal{A}\}\right|,$$

where

$$I_m=[m-\lambda_m+1,m]\ \text{ and }\ I_n=[n-\mu_n+1,n]\ \text{ and }\ \lambda_{m,n}=\lambda_m\mu_n,$$

is called the $\lambda-$density of $\mathcal{A}$ provided that the limit exists, (see [47]).

We refer to $(k\in I_m,l\in I_n)$ throughout the paper as $(k,l)\in I_{m,n}=I_m\times I_n$.

Definition 2.7. A double sequence $z=(z_{k,l})$ is said to be $f-$statistically convergent to a point $\mathcal{L}$ of Z if, for every neighbourhood U of 0,

$$\lim_{m,n\to\infty}\frac{1}{f\left(mn\right)}f\left(\left|\{k\leq m,l\leq n:z_{k,l}-\mathcal{L}\notin U\}\right|\right)=0$$

where f is unbounded modulus function. Here, $S_f^2(Z)-\lim_{k,l\to\infty}z_{k,l}=\mathcal{L}$ is written, and $S_f^2(Z)$ is used to represent the set of all $f-$statistically convergent double sequences, (see [46]).

Lemma 2.8. *Let f be a modulus function and let $0<\sigma<1$. Then, we obtain $f(z)\leq 2f(1)\sigma^{-1}z$ for any $z\geq\sigma$, (see [36]).*

3. Main Results

The notions of $f_\lambda-$statistical convergence and $f_\lambda-$statistical Cauchy for double sequences in topological groups with an unbounded modulus function are defined in this section. Some inclusion theorems for different unbounded modulus functions are also examined in this section.

Definition 3.1. A double sequence $z=(z_{k,l})$ is called $S^2_{f_\lambda}(Z)-$convergent to a point $\mathcal{L}$ of Z if for each neighbourhood U of 0,

$$\delta^2_{f_\lambda}\left(\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}\right)=0$$

i.e.

$$\lim_{m,n\to\infty}\frac{1}{f\left(\lambda_{m,n}\right)}f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)=0$$

where f is unbounded modulus function. In this case, we write $S^2_{f_\lambda}(Z)-\lim\limits_{k,l\to\infty} z_{k,l}=\mathcal{L}$ and we denote the set of all $f_\lambda-$statistically convergent double sequences by $S^2_{f_\lambda}(Z)$.

Definition 3.2. A double sequence $z=(z_{k,l})$ is called $S^2_{f_\lambda}C(Z)$ sequence if for each neighbourhood U of 0, there exists $M=M(U)$ and $N=N(U)$ such that

$$\lim_{m,n\to\infty}\frac{1}{f\left(\lambda_{m,n}\right)}f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-z_{M,N}\notin U\}|\right)=0$$

where f is unbounded modulus function. We denote the set of all $f_\lambda-$Cauchy double sequences by $S^2_{f_\lambda}C(Z)$.

Theorem 3.3. *If a double sequence $z=(z_{k,l})$ is $S^2_{f_\lambda}(Z)-$convergent sequence in Z, then $S^2_{f_\lambda}(Z)-\lim\limits_{k,l} z_{k,l}$ is unique, where f is unbounded modulus function.*

Proof. Let $z=(z_{k,l})$ is $S^2_{f_\lambda}(Z)-$convergent sequence in Z. Suppose that $(z_{k,l})$ has two different $S^2_{f_\lambda}(Z)$ limits, $\mathcal{L}_1,\mathcal{L}_2$ say. Since Z is Hausdorff space there exists a neighbourhood U of 0 such that $\mathcal{L}_1-\mathcal{L}_2\notin U$. Also for each neighbourhood U of 0 there exists symmetric neighbourhood V of 0 such that $V+V\subset U$, (see [2]). Write $y_{k,l}=\mathcal{L}_1-\mathcal{L}_2$ for all $(k,l)\in\mathbb{N}\times\mathbb{N}$. Therefore for all $(m,n)\in\mathbb{N}\times\mathbb{N}$

$$\begin{aligned}&\{(k,l)\in I_{m,n}: y_{k,l}\notin U\}\\&\qquad\subset\{(k,l)\in I_{m,n}: \mathcal{L}_1-z_{k,l}\notin V\}\cup\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}_2\notin V\}.\end{aligned}$$

Now it follows from this inclusion that, for all $(m,n)\in\mathbb{N}\times\mathbb{N}$ and since f is increasing,

$$\begin{aligned}\tfrac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}: y_{k,l}\notin U\}|\right)&\leq\tfrac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}: \mathcal{L}_1-z_{k,l}\notin V\}|\right)\\&\quad+\tfrac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}_2\notin V\}|\right).\end{aligned}$$

Since $S^2_{f_\lambda}(Z)-\lim\limits_{k,l} z_{k,l}=\mathcal{L}_1$ and $S^2_{f_\lambda}(Z)-\lim\limits_{k,l} z_{k,l}=\mathcal{L}_2$, we get

$$\begin{aligned}\lim_{m,n\to\infty}\tfrac{1}{f(\lambda_{m,n})}&\quad f\left(|\{(k,l)\in I_{m,n}: y_{k,l}\notin U\}|\right)\\&\leq\lim_{m,n\to\infty}\tfrac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}: \mathcal{L}_1-z_{k,l}\notin V\}|\right)\\&\quad+\lim_{m,n\to\infty}\tfrac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}_2\notin V\}|\right).\end{aligned}$$

Hence $1\leq 0+0=0$. This contradiction shows that $\mathcal{L}_1=\mathcal{L}_2$. □

Theorem 3.4. *Suppose that $P-\lim\limits_{k,l} z_{k,l}=\mathcal{L}$ and $S^2_{f_\lambda}(Z)-\lim\limits_{k,l\to\infty} y_{k,l}=0$. Then $S^2_{f_\lambda}(Z)-\lim\limits_{k,l\to\infty}(z_{k,l}+y_{k,l})=\mathcal{L}$, where f is unbounded modulus function.*

Theorem 3.5. *Let $z=(z_{k,l})$ and $y=(y_{k,l})$ be two double sequences in Z. If $S^2_{f_\lambda}(Z)-\lim\limits_{k,l\to\infty} z_{k,l}=\mathcal{L}_1$ and $S^2_{f_\lambda}(Z)-\lim\limits_{k,l\to\infty} y_{k,l}=\mathcal{L}_2$, then $S^2_{f_\lambda}(Z)-\lim\limits_{k,l\to\infty}(z_{k,l}+y_{k,l})=\mathcal{L}_1+\mathcal{L}_2$, where f is unbounded modulus function.*

Theorem 3.6. *Let $z=(z_{k,l})$ be a double sequence in Z and f be an unbounded modulus function. If*

$$\liminf_{m,n\to\infty}\frac{mn}{f\left(\lambda_{m,n}\right)}<\infty,$$

then $S^2_{f_\lambda}(Z)-$convergent sequence is $S^2_f(Z)-$convergent sequence, i.e. $S^2_f(Z)\subset S^2_{f_\lambda}(Z)$.

Proof. Suppose that $\liminf\limits_{m,n\to\infty}\frac{mn}{f(\lambda_{m,n})}<\infty$ and $z=(z_{k,l})$ be $S^2_f(Z)-$convergent sequence. Let U be any neighbourhood of 0. Then for all $(m,n)\in\mathbb{N}\times\mathbb{N}$,

$$\{k\leq m,l\leq n:z_{k,l}-\mathcal{L}\notin U\}\supset\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right)\\
&\leq\frac{1}{f(\lambda_{m,n})}f\left(|\{k\leq m,l\leq n:z_{k,l}-\mathcal{L}\notin U\}|\right)\\
&\leq\frac{mn}{f(\lambda_{m,n})}\frac{f(mn)}{mn}\frac{1}{f(mn)}f\left(|\{k\leq m,l\leq n:z_{k,l}-\mathcal{L}\notin U\}|\right)\\
&\leq\frac{mn}{f(\lambda_{m,n})}f(1)\frac{1}{f(mn)}f\left(|\{k\leq m,l\leq n:z_{k,l}-\mathcal{L}\notin U\}|\right).
\end{aligned}$$

Taking the limit as $m,n\to\infty$, we get $S^2_f(Z)\subset S^2_{f_\lambda}(Z)$. □

Theorem 3.7. *Let $z=(z_{k,l})$ be a double sequence in Z and f be an unbounded modulus function. If*

$$\lim_{m,n\to\infty}\frac{mn}{f\left(\lambda_{m,n}\right)}=1\ \textit{and}\ \lim_{m,n\to\infty}\frac{f\left(mn-\lambda_{m,n}\right)}{mn}=0,$$

then $S^2_f(Z)-$convergent sequence is $S^2_{f_\lambda}(Z)-$convergent sequence, i.e. $S^2_{f_\lambda}(Z)\subset S^2_f(Z)$.

Proof. Suppose that $\lim\limits_{m,n\to\infty}\frac{mn}{f(\lambda_{m,n})}=1$ and $\lim\limits_{m,n\to\infty}\frac{f(mn-\lambda_{m,n})}{mn}=0$. Let U be any neighbourhood of 0. Since $\lambda_{m,n}\leq mn\Rightarrow f\left(\lambda_{m,n}\right)\leq f\left(mn\right)$, we get

$$\begin{aligned}
&\frac{1}{f(mn)}f\left(|\{k\leq m,l\leq n:z_{k,l}-\mathcal{L}\notin U\}|\right)\\
&=\frac{1}{f(mn)}f\left(|\{k\leq m-\lambda_m,l\leq n-\lambda_n:z_{k,l}-\mathcal{L}\notin U\}|\right)\\
&+\frac{1}{f(mn)}f\left(|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right)\\
&\leq\frac{f\left(mn-\lambda_{m,n}\right)}{f(mn)}+\frac{1}{f(mn)}f\left(|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right)\\
&\leq\frac{f\left(mn-\lambda_{m,n}\right)}{mn}\frac{mn}{f\left(\lambda_{m,n}\right)}\frac{f\left(\lambda_{m,n}\right)}{f(mn)}+\frac{1}{f\left(\lambda_{m,n}\right)}f\left(|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right).
\end{aligned}$$

Taking the limit as $m,n\to\infty$, we get $S^2_{f_\lambda}(Z)\subset S^2_f(Z)$. □

Theorem 3.8. *Let $z=(z_{k,l})$ be a double sequence in Z and $\lambda=(\lambda_{m,n})$ and $\varphi=(\varphi_{m,n})$ be two sequences of positive numbers such that $\lambda_{m,n}\leq\varphi_{m,n}$ for all $(m,n)\in\mathbb{N}\times\mathbb{N}$.*

i) *if* $\liminf_{m,n\to\infty}\frac{f(\lambda_{m,n})}{f(\varphi_{m,n})}>0$, *then* $S^2_{f_\varphi}(Z)\subset S^2_{f_\lambda}(Z)$,

ii) *if* $\lim_{m,n\to\infty}\frac{\varphi_{m,n}}{\lambda_{m,n}}=1$, *then* $S^2_{f_\lambda}(Z)\subset S^2_{f_\varphi}(Z)$,

where f is unbounded modulus function.

Proof. i) Suppose that $\lambda_{m,n}\leq\varphi_{m,n}$ for all $(m,n)\in\mathbb{N}\times\mathbb{N}$, which implies that $f(\lambda_{m,n})\leq f(\varphi_{m,n})$. Assume that $\liminf_{m,n\to\infty}\frac{f(\lambda_{m,n})}{f(\varphi_{m,n})}>0$ are satisfied. Since $\lambda_{m,n}\leq\varphi_{m,n}$ for all $(m,n)\in\mathbb{N}\times\mathbb{N}$, so $I_{m,n}\subset J_{m,n}$, for a neighbourhood U of 0, we get

$$\{(k,l)\in J_{m,n}:z_{k,l}-\mathcal{L}\notin U\}\supset\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}.$$

Therefore, we can write

$$\begin{aligned}&\frac{1}{f(\varphi_{m,n})}f\left(|\{(k,l)\in J_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right)\\&\geq\frac{f(\lambda_{m,n})}{f(\varphi_{m,n})}\frac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right)\end{aligned}$$

for all $(m,n)\in\mathbb{N}\times\mathbb{N}$. Since $\liminf_{m,n\to\infty}\frac{f(\lambda_{m,n})}{f(\varphi_{m,n})}>0$ and also taking as $m,n\to\infty$, we get $S^2_{f_\varphi}(Z)\subset S^2_{f_\lambda}(Z)$.

ii) Let $S^2_{f_\lambda}(Z)-\lim_{k,l}z_{k,l}=\mathcal{L}$, for the neighbourhood U of 0. Since $I_{m,n}\subset J_{m,n}$, we can write

$$\begin{aligned}&\frac{1}{\varphi_{m,n}}\left|\{(k,l)\in J_{m,n}:z_{k,l}-\mathcal{L}\notin U\}\right|\\&=\frac{1}{\varphi_{m,n}}\left|\{m-\varphi_m+1<k<m-\lambda_m,n-\varphi_n+1<l<n-\lambda_n:z_{k,l}-\mathcal{L}\notin U\}\right|\\&+\frac{1}{\varphi_{m,n}}\left|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}\right|\\&\leq\frac{\varphi_{m,n}-\lambda_{m,n}}{\varphi_{m,n}}+\frac{1}{\varphi_{m,n}}\left|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}\right|\\&\leq\frac{\varphi_{m,n}-\lambda_{m,n}}{\lambda_{m,n}}+\frac{1}{\lambda_{m,n}}\left|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}\right|\\&\leq\left(\frac{\varphi_{m,n}}{\lambda_{m,n}}-1\right)+\frac{1}{\lambda_{m,n}}\left|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}\right|,\end{aligned}$$

for all $(m,n)\in\mathbb{N}\times\mathbb{N}$. Hence, using Lemma 2.8, we get

$$\begin{aligned}&\frac{1}{f(\varphi_{m,n})}f\left(|\{(k,l)\in J_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right)\\&\leq\left(\frac{\varphi_{m,n}}{\lambda_{m,n}}-1\right)+\frac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right)\\&\leq 2f(1)\sigma^{-1}\left(\frac{\varphi_{m,n}}{\lambda_{m,n}}-1\right)+\frac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right).\end{aligned}$$

Since $\lim_{m,n\to\infty}\frac{\varphi_{m,n}}{\lambda_{m,n}}=1$ by *(ii)* the first terms goes to 0 since $S^2_{f_\lambda}(Z)-\lim_{k,l} z_{k,l}=\mathcal{L}$, the second term of right hand side of above inequality tends to 0 as $m,n\to\infty$. This implies that $S^2_{f_\lambda}(Z)\subset S^2_{f_\varphi}(Z)$.

□

Theorem 3.9. *Let $z=(z_{k,l})$ be a double sequence in Z and f and g be two unbounded modulus functions. Then*

i) *if $\lim_{s\to\infty}\frac{f(s)}{g(s)}>0$, then a sequence $(z_{k,l})$ is $f_\lambda-$statistically convergent double sequence in Z if it is $g_\lambda-$statistically convergent double sequence in Z, that is $S^2_{g_\lambda}(Z)\subset S^2_{f_\lambda}(Z)$,*

ii) *if $\lim_{s\to\infty}\frac{f(s)}{g(s)}<\infty$, then a sequence $(z_{k,l})$ is $g_\lambda-$statistically convergent double sequence in Z if and only if it is $f_\lambda-$statistically convergent double sequence in Z, that is $S^2_{g_\lambda}(Z)=S^2_{f_\lambda}(Z)$.*

Proof. Let f and g be two unbounded modulus functions.

i) Suppose $\lim_{s\to\infty}\frac{f(s)}{g(s)}=\alpha>0$, then $\lim_{m,n\to\infty}\frac{f(\lambda_{m,n})}{g(\lambda_{m,n})}=\alpha>0$ and $(z_{k,l})$ is g_λ-statistically convergent double sequence to $\mathcal{L}$, that is $S^2_{g_\lambda}(Z)-\lim_{k,l} z_{k,l}=\mathcal{L}$. Then given any $\varepsilon>0$ there exist a real number r_0 such that $(\alpha-\varepsilon)\,g(\lambda_{m,n})<f(\lambda_{m,n})<(\alpha+\varepsilon)\,g(\lambda_{m,n})$ if $m>m_0$ and $n>n_0$. Therefore we have the inequality $f(\lambda_{m,n})<2\alpha g(\lambda_{m,n})$ if $m>m_0$ and $n>n_0$. Now we may write the inequality

$$\frac{1}{g(\lambda_{m,n})}g\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)$$
$$\geq\frac{1}{2\alpha}\frac{f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)}{f(\lambda_{m,n})}\frac{f(\lambda_{m,n})}{g(\lambda_{m,n})}$$

for each neighbourhood U of 0, if $|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|>\max\{m_0,n_0\}$. Since $\lim_{m,n\to\infty}\frac{f(\lambda_{m,n})}{g(\lambda_{m,n})}=\alpha>0$ from the above inequality we obtain

$$\lim_{m,n\to\infty}\frac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)=0$$

if

$$\lim_{m,n\to\infty}\frac{1}{g(\lambda_{m,n})}g\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)=0.$$

Therefore $S^2_{g_\lambda}(Z)\subset S^2_{f_\lambda}(Z)$.

ii) We may write the following equality for each neighbourhood U of 0

$$\frac{1}{g(\lambda_{m,n})}g\left(|\{(k,l)\in I_{m,n}: z(k,l)-\mathcal{L}\notin U\}|\right)$$
$$=\frac{g\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)}{f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)}\frac{f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)}{f(\lambda_{m,n})}\frac{f(\lambda_{m,n})}{g(\lambda_{m,n})}.$$

Suppose $\lim_{m,n\to\infty}\frac{f(\lambda_{m,n})}{g(\lambda_{m,n})}=\alpha<\infty$ and so that $\lim_{m,n\to\infty}\frac{g(\lambda_{m,n})}{f(\lambda_{m,n})}=\frac{1}{\alpha}$. Using this fact from the above equality, we obtain

$$\lim_{m,n\to\infty}\frac{1}{f(\lambda_{m,n})}f\left(|\{(k,l)\in I_{m,n}: z_{k,l}-\mathcal{L}\notin U\}|\right)=0$$

if and only if

$$\lim_{m,n\to\infty}\frac{1}{g(\lambda_{m,n})}g\left(|\{(k,l)\in I_{m,n}:z_{k,l}-\mathcal{L}\notin U\}|\right)=0.$$

Therefore $S^2_{g_\lambda}(Z)=S^2_{f_\lambda}(Z)$.

□

Theorem 3.10. *Let $z=(z_{k,l})$ be a double sequence in Z and f and g be two unbounded modulus functions. Then*

i) *if $\lim_{s\to\infty}\frac{f(s)}{g(s)}>0$, then a sequence $(z_{k,l})$ is $f_\lambda-$Cauchy double sequence in Z if it is $g_\lambda-$Cauchy double sequence in Z,*

ii) *if $\lim_{s\to\infty}\frac{f(s)}{g(s)}<\infty$, then a sequence $(z_{k,l})$ is $g_\lambda-$Cauchy double sequence in Z if and only if it is $f_\lambda-$Cauchy double sequence in Z.*

Proof. The proof can be similarly to the proof of Theorem 3.9. □

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